On the spectral gap and the expansion of random simplicial complexes

Nikolaos Fountoulakis
University of Birmingham

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The graph Laplace operator

Given a graph $G = (V, E)$, the (combinatorial) Laplace operator is defined as

$$\Delta_G = D_G - A_G,$$

where $D_G = \mathrm{diag}(\deg(v))_{v \in V}$ and A_G is the adjacency matrix of G.

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Alon and Milman (1985) showed that for any non-empty (proper) subset $A \subset V(G)$ we have

$$\lambda(G) \leq \frac{n \cdot e(A, V \setminus A)}{|A||V \setminus A|} =: h(A; G).$$

The graph Laplace operator

Setting $h(G) = \min_{A\,:\,0<|A|\leq|V|/2} h(A;G)$ one can complete the above inequality with a lower bound and get

$$\frac{h^2(G)}{8d_{\max}(G)} \leq \lambda(G) \leq h(G), \quad \text{(Cheeger inequalities)}$$

where $d_{\max}(G)$ is the maximum degree of G.

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where $d_{\max}(G)$ is the maximum degree of G.
In particular, the above inequality implies that

$$\lambda(G) \leq \frac{|V(G)|}{|V(G)|-1} d_{\min}(G),$$

where $d_{\min}(G)$ is the minimum degree of G.

The Laplace operator of a d-dimensional simplicial complex

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We will consider the set of r-forms $\Omega^{(r)}$: all functions $f : \mathcal{K}_{\pm}^{(r)} \to \mathbb{R}$ which are anti-symmetric:

$$f(\sigma) = -f(\bar{\sigma}), \text{ for any } \sigma \in \mathcal{K}_{\pm}^{(r)}.$$

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The boundary operator $\partial_j : \Omega^{(j)} \to \Omega^{(j-1)}$:

for $f \in \Omega^{(j)}$ and $\sigma \in \mathcal{K}_{\pm}^{(j-1)}$

$$(\partial_j f)(\sigma) = \sum_{v : v\sigma \in \mathcal{K}_{\pm}^{(j)}} f(v\sigma),$$

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The coboundary operator $\delta_j : \Omega^{(j)} \to \Omega^{(j+1)}$:

for $f \in \Omega^{(j)}$ and $\sigma = [v_0, \ldots, v_{j+1}]$

$$(\delta_j f)(\sigma) = \sum_{i=0}^{j+1} (-1)^i f(\sigma \setminus v_i).$$

The Laplace operator of a d-dimensional simplicial complex

The Laplace operator $\Delta : \Omega^{(d-1)} \to \Omega^{(d-1)}$ of $\mathcal{K}$ is defined as

$$\Delta = \Delta^{+} + \Delta^{-},$$

where

$$\Delta^{+} = \partial_d \delta_{d-1} \quad \text{(the upper Laplacian)},$$

and

$$\Delta^{-} = \delta_{d-2} \partial_{d-1} \quad \text{(the lower Laplacian)}.$$

The spectrum of Δ

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Thus, with $Z_{d-1} = \mathrm{Ker}\partial_{d-1}$ we have

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In what follows, we focus on the spectrum of $\Delta^{+} = \partial_d \delta_{d-1}$.

The Laplace operator of a d-dimensional simplicial complex

The upper Laplacian can be written as follows: for $f \in \Omega^{(d-1)}$ and $\sigma = [v_0, \ldots, v_{d-1}] \in Y_{\pm}^{(d-1)}$ we have

$$(\Delta^+ f)(\sigma) = \deg(\sigma) f(\sigma) - \sum_{v : v\sigma \in \mathcal{K}_{\pm}^{(d)}} \sum_{i=0}^{d-1} (-1)^i f(v\sigma \setminus v_i),$$

where $\deg(\sigma)$ is the co-degree of σ in $\mathcal{K}$:

$$\deg(\sigma) = |d\text{-faces containing } \sigma|.$$

The spectral gap of Δ^+ and a Cheeger inequality

We define

$$h(\mathcal{K}) = \min_{|\mathcal{K}^{(0)}| = A_0 \uplus \cdots \uplus A_d} \frac{n \cdot |F(A_0, \ldots, A_d)|}{\prod_{i=0}^{d} |A_i|}, \qquad (1)$$

where the minimum is taken over all partitions of $\mathcal{K}^{(0)} = V$ into $d+1$ non-empty parts $A_0, \ldots, A_d$ and $F(A_0, \ldots, A_d)$ is the set of d-faces with exactly one vertex in each one of the parts.

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The following theorem was proved by Parzanchevski, Rosenthal, and Tessler (2016):

Theorem

For a finite complex $\mathcal{K}$ with a complete skeleton,

$$\lambda(\mathcal{K}) \leq h(\mathcal{K}).$$

The spectral gap of Δ^+ and a Cheeger inequality

One would hope that the would be completed into

$$\frac{h(\mathcal{K})^2}{d_{\max}(\mathcal{K})} \leq \lambda(\mathcal{K}) \leq h(\mathcal{K}),$$

but this is NOT true.

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but this is NOT true.

Instead, Parzanchevski, Rosenthal, and Tessler conjecture that

$$\frac{h(\mathcal{K})^2}{C} - c \leq \lambda(\mathcal{K}),$$

where C depends on the maximum co-degree of $\mathcal{K}$.

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1. all possible faces of dimension up to $d - 1$ are present,
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2. each subset of $[n]$ of size $d+1$ becomes a d-face with probability $p = p(n) \in [0,1]$, independently of every other subset of size $d+1$.

For $d = 1$, this includes the binomial random graph $G(n,p)$.

The Linial-Meshulam random complex

Linial and Meshulam showed that $Y(n,p;2)$ undergoes a sharp transition which generalises the connectivity transition of $G(n,p)$.

Theorem (Linial and Meshulam 2006)

Let $H^1(Y(n,p;2);\mathbb{Z}_2)$ be the first cohomology group of $Y(n,p;2)$. Then

$$\lim_{n\to\infty} \mathbb{P}\left(H^1(Y(n,p;d);\mathbb{Z}_2) \text{ is trivial}\right) = \begin{cases} 1, & \text{if } p = \frac{2\log n + \omega(n)}{n} \\ 0, & \text{if } p = \frac{2\log n - \omega(n)}{n} \end{cases},$$

where $\omega(n) \to \infty$ as $n \to \infty$.

The Linial-Meshulam random complex

This was generalised in a number of papers.

Theorem (Meshulam and Wallach 2008, Gundert and Wagner, 2016)

Let $H^{d-1}(Y(n,p;d);R)$ be the first cohomology group of $Y(n,p;d)$. Then

$$\lim_{n\to\infty} \mathbb{P}\left(H^1(Y(n,p;d);R) \text{ is trivial}\right) = \begin{cases} 1, & \text{if } p = \frac{d\log n+\omega(n)}{n} \\ 0, & \text{if } p = \frac{d\log n-\omega(n)}{n} \end{cases},$$

where $\omega(n)\to\infty$ as $n\to\infty$.

The cohomology group H^{d-1}

This define over the set of $d-1$-forms $\Omega^{(d-1)}$ as

$$H^{d-1}(\mathcal{K}) = \mathrm{Ker}\delta_{\mathrm{d-1}}/\mathrm{Im}\delta_{d-2}.$$

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When $np = d \log n - \omega(n)$, then w.h.p. there is a $d-1$-face of co-degree 0.

The spectral gap and the Cheeger constant of $Y(n,p;d)$

Let $\delta(Y(n,p;d))$ be the minimum co-degree among the $d-1$-faces of $Y(n,p;d)$. We showed the following.

Theorem (F. and Przykucki, 2020+)

For $d \geq 2$, let $p = \frac{(1+\varepsilon) d \log n}{n}$, where $\varepsilon > 0$ is fixed. There exists $C > 0$ such that w.h.p.

$$\begin{aligned} \delta(Y(n,p;d)) - C\sqrt{\log n} &\leq \lambda(Y(n,p;d)) \leq h(Y(n,p;d)) \\ &\leq (1 + O(1/n))\delta(Y(n,p;d)). \end{aligned}$$

Furthermore, w.h.p.

$$|\delta(Y(n,p;d)) - (1+\varepsilon) a d \log n| < C\sqrt{\log n},$$

where $a = a(\varepsilon)$ is the solution to $\varepsilon = (1+\varepsilon)(1 - \log a)a$.

The spectral gap and the Cheeger constant of $Y(n, p; d)$

The minimum co-degree

This proof uses large deviations estimates for the binomial distribution and a first moment argument - it extends an argument of Kolokolnikov, Osting and von Brecht (2014) about the minimum degree of $G(n, p)$ in the corresponding regime.

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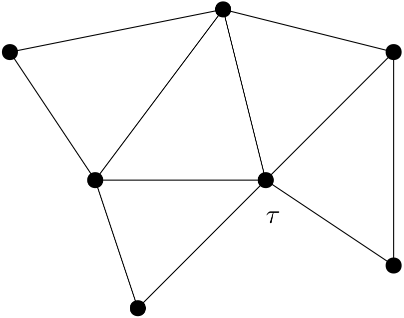
The inequality $h(Y(n,p;d)) \leq (1+O(1/n))\delta(Y(n,p;d))$ follows from taking the partition:
if $\sigma = \{a_0,\ldots,a_{d-1}\}$ is a $d-1$-face with the minimum co-degree, then taking $A_i = \{a_i\}$ for $0 \leq i \leq d-1$ and $A_d = [n] \setminus A$ gives us a partition with $|F(A_0, A_1, \ldots, A_d)| = \delta(Y(n,p;d))$. Thus,

$$\begin{aligned} h(Y(n,p;d)) &= \min_{[n]=A_0 \uplus \cdots \uplus A_d} \frac{n \cdot |F(A_0,\ldots,A_d)|}{\prod_{i=0}^{d} |A_i|} \\ &\leq (1+O(1/n))\delta(Y(n,p;d)). \end{aligned}$$

The link graph

Figure: The link graph of face τ in a 2-dimensional complex



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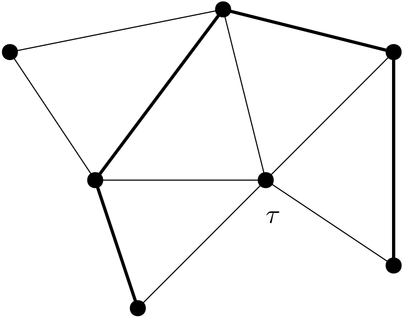

The sprectral gap of $Y(n, p; d)$ - lower bound

Lower bound on $\lambda(\Delta^+)$

We will show that w.h.p.

$$\inf_{0 \neq f \in Z_{d-1}} \frac{\langle \Delta^+ f, f \rangle}{\langle f, f \rangle} \geq \delta(Y(n,p;d)) - O(\sqrt{\log n}).$$

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We use the following decomposition of the Laplace operator of a complex X that is due to Garland 1973

$$\Delta^+ = \sum_{\tau \in X^{(d-2)}} \Delta_\tau^+ - (d-1)D,$$

where $(Df)(\sigma) = \mathrm{deg}(\sigma) f(\sigma)$ and

$$\Delta_\tau^+ = D_{\mathrm{lk}\tau} - A_{\mathrm{lk}\tau}$$

is the Laplace operator of the link graph $\mathrm{lk}\tau$ of τ in X.

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We show that for any $f \in Z_{d-1}$ we can write:

$$\langle \Delta^+ f, f \rangle = \sum_{\tau \in X^{(d-2)}} \left(\frac{1}{d} \langle D_{\mathrm{lk}\tau} f_\tau, f_\tau \rangle - \langle A_{\mathrm{lk}\tau} f_\tau, f_\tau \rangle \right),$$

where

$$f_\tau : (\mathrm{lk}\tau)^{(0)} \to \mathbb{R} \text{ as } f_\tau(v) = f(v\tau).$$

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We show that

$$\frac{1}{d} \sum_{\tau \in X^{(d-2)}} \langle D_{\mathrm{lk}\tau} f_\tau, f_\tau \rangle \geq \delta(X) \langle f, f \rangle.$$

The spectral gap of $Y(n, p; d)$

Now, if $X = Y(n, p; d)$, then for any $\tau \in Y^{(d-2)}(n, p; d)$ the link graph $\mathrm{lk}\tau$ is a random graph distributed as $G(n, p)$. Hence, $A_{\mathrm{lk}\tau}$ is the adjacency matrix of a $G(n, p)$ distributed random graph.

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Also, the assumption that $f \in Z_{d-1} = \mathrm{Ker}\partial_{d-1}$ translates into $< f_\tau, \mathbf{1} >= 0$, for all $\tau \in Y^{(d-2)}$.

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These random graphs are not independent, but a result of Feige and Ofek 2005 implies that with probability $1 - o(n^{-(d-1)})$ we have

$$\langle A_{\mathrm{lk}\tau} f_\tau, f_\tau \rangle \leq C\sqrt{np} = O(\sqrt{\log n}).$$

The spectral gap of $Y(n, p; d)$

The union bound over all $O(n^{d-1})$ choices of $\tau \in Y^{(d-2)}(n, p; d)$ implies that w.h.p. for all $f \in Z_{d-1}$

$$\langle \Delta^{+} f, f \rangle \geq \left(\delta(Y(n, p; d)) - O(\sqrt{\log n}) \right) \langle f, f \rangle.$$

Random walks on $Y(n, p; d)$

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Let $Y = Y(n,p;d)$. We consider a random walk on $Y^{(d-1)}$. For distinct $\sigma, \sigma' \in Y^{(d-1)}$, we write $\sigma \sim \sigma'$, if there exists $\rho \in Y^{(d)}$ such that $\sigma, \sigma' \subset \rho$.

Random walks on $Y(n,p;d)$

Let $Y = Y(n,p;d)$. We consider a random walk on $Y^{(d-1)}$.
For distinct $\sigma, \sigma' \in Y^{(d-1)}$, we write $\sigma \sim \sigma'$, if there exists $\rho \in Y^{(d)}$ such that $\sigma, \sigma' \subset \rho$.
If $(X_0, X_1, \ldots)$ denotes this Markov chain, then for any $n \geq 1$ the transition probabilities are

$$\mathbb{P}\left(X_n = \sigma' \mid X_{n-1} = \sigma\right) = \begin{cases} \frac{1}{d \cdot \deg(\sigma)} & \text{if } \sigma \sim \sigma' \\ 0 & \text{otherwise} \end{cases}.$$

Random walks on $Y(n, p; d)$

In a more general setting, one may consider a γ-*lazy* version of this random walk, for $\gamma \in (0, 1)$, where

$$\mathbb{P}\left(X_n = \sigma \mid X_{n-1} = \sigma\right) = \gamma$$

and for $\sigma \sim \sigma'$

$$\mathbb{P}\left(X_n = \sigma' \mid X_{n-1} = \sigma\right) = \frac{1-\gamma}{d \cdot \deg(\sigma)}.$$

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The stationary distribution on $Y^{(d-1)}$, denoted by π, is such that $\pi(\sigma)$ for any $\sigma \in Y^{(d-1)}$ we have

$$\pi(\sigma) = \frac{\deg(\sigma)}{(d+1) \cdot |Y^{(d)}|}.$$

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For any non-empty subset $S \subset Y^{(d-1)}$ we define

$$\Phi_Y(S) = \frac{Q(S, \overline{S})}{\pi(S)},$$

where $\overline{S} = Y^{(d-1)} \setminus S$ and
$Q(S, \overline{S}) = \sum_{\sigma \in S} \sum_{\sigma' \in \overline{S} : \sigma' \sim \sigma} \pi(\sigma) \cdot \frac{1}{d \cdot \deg(\sigma)}$ and
$\pi(S) = \sum_{\sigma \in S} \pi(\sigma)$.

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The conductance Φ_Y is defined as

$$\Phi_Y = \min_{S \subset Y^{(d-1)}: 0 < \pi(S) \leq 1/2} \Phi_Y(S).$$

Random walks on $Y(n,p;d)$

We prove that w.h.p. Φ_Y is bounded away from 0.

Theorem (F. and Przykucki 2020+)

Let $Y = Y(n,p;d)$ where $np = (1+\varepsilon)d\log n$ and $\varepsilon > 0$ is fixed. There exists $\delta > 0$ such that w.h.p.

$$\Phi_Y > \delta.$$

The conductance of $Y(n,p;d)$

Given $S \subset Y^{(d-1)}$ let

$$\partial^+ S = \{\rho \subset Y^{(d)} \ : \ \text{there exists } \ \sigma \in S \text{ such that } \sigma \subset \rho\}.$$

$$\text{and } \ B_S = \{\sigma \in \partial^+ S : \partial\sigma \subset S\}$$

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We show that

$$Q(S,\overline{S}) \geq \frac{1}{d(d+1)\cdot |Y^{(d)}|} \cdot |\partial^+ S \setminus B_S|.$$

and

$$\pi(S) = \frac{\sum_{\sigma\in S} \deg(\sigma)}{d\cdot |Y^{(d)}|} \leq \frac{d\cdot |\partial^+ S|}{(d+1)\cdot |Y^{(d)}|} < \frac{|\partial^+ S|}{|Y^{(d)}|},$$

whereby

The conductance of $Y(n, p; d)$

Given $S \subset Y^{(d-1)}$ let

$$\partial^+ S = \{\rho \subset Y^{(d)} \ : \text{ there exists } \ \sigma \in S \text{ such that } \sigma \subset \rho\}.$$

$$\text{and} \quad B_S = \{\sigma \in \partial^+ S : \partial\sigma \subset S\}$$

$$\Phi_Y \geq \frac{1}{d(d+1)} \cdot \min_{S \subset Y^{(d-1)} : 0 < \pi(S) \leq \frac{1}{2}} \frac{|\partial^+ S \setminus B_S|}{|\partial^+ S|}.$$

The conductance of $Y(n, p; d)$

We apply a union bound in order to bound from below $\frac{|\partial^+ S \setminus B_S|}{|\partial^+ S|}$ for all $S \subset Y^{(d-1)}$ with $0 < \pi(S) \leq \frac{1}{2}$.

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Given $S \subset Y^{(d-1)}$ and $1 \leq i \leq d+1$, let

$$F_i(S) = \{\rho \in \partial^+ S : |\partial \rho \cap S| = i\},$$

and set $f_i(S) = |F_i(S)|$.

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Denoting $|S| = m$, by double counting, we have that

$$\sum_{i=1}^{d+1} i f_i(S) = m(n-d).$$

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Note that

$$f_{d+1}(S) = K_{d+1}^{(d)}(S),$$

in other words...

$$f_{d+1}(S) = |B_S|.$$

The conductance of $Y(n, p; d)$

To bound $K^{(d)}_{d+1}(S)$ we use a weaker form of the Kruskal-Katona theorem.

Theorem

Suppose $r \geq 1$ and G is an r-uniform hypergraph with

$$m = \binom{x_m}{r} = \frac{x_m(x_m - 1)\ldots(x_m - r + 1)}{r!}$$

hyperedges, for some real number $x_m \geq r$. Then $K^{(r)}_{r+1}(G) \leq \binom{x_m}{r+1}$, with equality if and only if x_m is an integer and $G = K^{(r)}_{x_m}$.

The conductance of $Y(n, p; d)$

This implies that

$$\frac{f_{d+1}(S)}{nm} \leq \frac{1}{n} \frac{\binom{x_m}{d+1}}{\binom{x_m}{d}} = \frac{x_m - d}{n(d+1)} \leq \frac{(md!)^{1/d}}{n(d+1)}.$$

The conductance of $Y(n,p;d)$

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Since $\sum_{i=1}^{d+1} i f_i(S) = m(n-d)$, we obtain

$$\sum_{i=1}^{d} f_i(S) \ge \frac{nm}{d}\left(1 - \frac{d}{n} - \frac{(md!)^{1/d}}{n}\right) = \frac{nm}{d}\left(1 - \frac{(md!)^{1/d}}{n} - o(1)\right).$$

The conductance of $Y(n, p; d)$

Thus, for example, if $m = o(n^d)$, then

$$|(\partial^+ S \setminus B_S) \cap Y^{(d)}| \sim \mathrm{Bin}\left(\frac{nm}{d}(1 - o(1)), p\right).$$

which is concentrated around $(1+\varepsilon)m \log n$, since $np = (1+\varepsilon)d \log n$,

The conductance of $Y(n,p;d)$

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which is concentrated around $(1+\varepsilon)m \log n$, since $np = (1+\varepsilon)d \log n$, and

$$|B_S| = f_{d+1}(S) \leq mn\frac{(md!)^{1/d}}{(d+1)n} = o(nm),$$

whereby

$$|B_S \cap Y^{(d)}| \sim \mathrm{Bin}\left(o(nm), p\right)$$

which is concentrated around $o(m \log n)$.

The conductance of $Y(n, p; d)$

So for a suitable $k_0 = o(nm)$ we have

$$\begin{aligned}
&\mathbb{P}\left(|(\partial^+ S \setminus B_S) \cap Y^{(d)}| \geq |\partial^+ S|/2\right) \\
&\quad \geq \mathbb{P}\left(|B_S \cap Y^{(d)}| \leq k_0 \text{ and } |(\partial^+ S \setminus B_S) \cap Y^{(d)}| \geq k_0\right) \\
&\quad \geq 1 - \exp\left(-(1 - o(1))(1 + \varepsilon) m \log n\right).
\end{aligned}$$

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$$\begin{aligned}\mathbb{P}&\left(|(\partial^+ S \setminus B_S) \cap Y^{(d)}| \geq |\partial^+ S|/2\right)\\ &\geq \mathbb{P}\left(|B_S \cap Y^{(d)}| \leq k_0 \text{ and } |(\partial^+ S \setminus B_S) \cap Y^{(d)}| \geq k_0\right)\\ &\geq 1 - \exp\left(-(1-o(1))(1+\varepsilon) m \log n\right).\end{aligned}$$

But the number of tightly connected sets of size m is estimated to being at most

$$4^m (dn)^m = \exp((1+o(1)) m \log n).$$

and the union bound works in this case...

Thank you!