Rainbow matchings in k-partite hypergraphs

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Conjecture (Aharoni, Howard)

Given s families $\mathcal{F}_1, \dots, \mathcal{F}_s \subset [n]^k$, such that

$$|\mathcal{F}_i| > (s-1)n^{k-1}, \quad i \in [s]$$

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Aharoni, Howard, 2017: True for $k = 2, 3$.
Lu, Yu, 2018: True for $n > 3(s-1)(k-1)$.

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Remark: if we replace $[n]^k$ with $\binom{[n]}{k}$, we will get (rainbow) Erdős matching conjecture.

Also note that $[n]^k \subset \binom{[nk]}{k}$.

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Thus, for some choice of $\mathcal{M}$, we have

$$|\mathcal{F} \cap \mathcal{M}| \geqslant s$$

Concentration

Fix $\mathcal{F} \subset [n]^k$, $|\mathcal{F}| = \alpha n^k$.
Consider a random perfect matching $\mathcal{M}$ in $[n]^k$, put $\eta := |\mathcal{F} \cap \mathcal{M}|$.

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Theorem (K. and Kupavksii)

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Corollary

Any $\mathcal{F}_1, \dots, \mathcal{F}_s \subset [n]^k$ such that

$$|\mathcal{F}_i| > (s + 4\sqrt{n \log n})n^{k-1}, \quad i \in [s],$$

contain a rainbow matching.

Concentration: proof outline

Assume $\mathcal{M} = (M_1, \ldots, M_n)$. We have $\eta = \eta_1 + \ldots + \eta_n$, where η_i indicates if $M_i \in \mathcal{F}$.

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Define an exposure martingale $X_0, \ldots, X_n$, where $X_i := \mathsf{E}[\eta \mid \eta_i, \ldots, \eta_1]$.

Note that $X_0 = \mathsf{E}[\eta] = \alpha n$ and $X_n = \eta$.

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For simplicity: $Y_i := \mathsf{E}[\eta_n \mid \eta_i, \ldots, \eta_1]$. Then $X_i = \sum_{j=1}^{i} \eta_j + (n-i)Y_i$.

$$|X_{i+1} - X_i| \leqslant 1 + (n-i-1)|Y_{i+1} - Y_i|.$$

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Azuma-Hoeffding inequality

If $X_0, \ldots, X_n$ is a martingale and $|X_i - X_{i-1}| \leqslant 2$ for any $i \in [n]$, then

$$\mathsf{P}\left[|X_n - X_0| \geqslant 2\beta\sqrt{n}\,\right] \leqslant 2e^{-\beta^2/2}$$

Proof of $|Y_{i+1} - Y_i| \leqslant 1/(n-i-1)$

For simplicity, assume that $i = 0$

For $i > 0$ we fix $M_1, \ldots, M_i$ and do the same for $[n-i]^k$

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Alon-Chung bound:

$$|Y_1 - Y_0| = \left|\frac{2e(\mathcal{F})}{\alpha} - \alpha\right| \leqslant \frac{|\lambda|(1-\alpha)}{d} = \frac{1}{n-1}.$$

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Average $\mathsf{E}\sum_{i=1}^{s} \Delta_i(\mathcal{M})$ is positive, so need to compensate by large positive deviations of the remaining i

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If $\mathcal{F}_1, \ldots, \mathcal{F}_s \subset [n]^k$ satisfy

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Answer: No!

Is there a natural correction?

Thank you!