Bi-uniform Property B

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Theoretical Computer Science
Jagiellonian Univeristy

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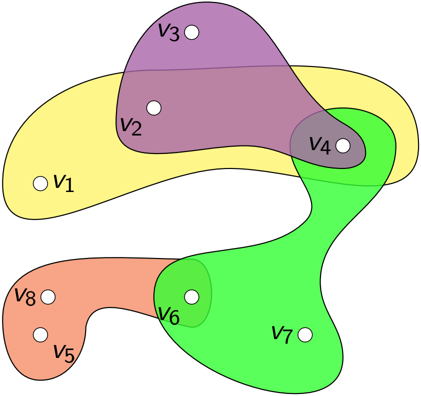
Hypergraphs and coloring



k-graph

every egde of size k

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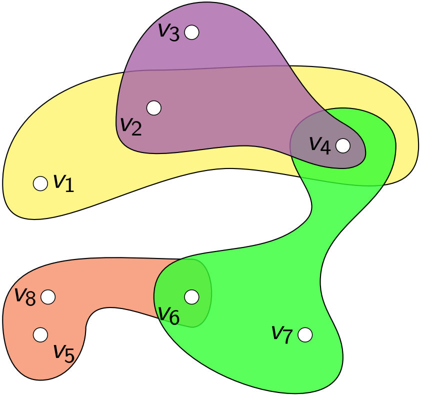

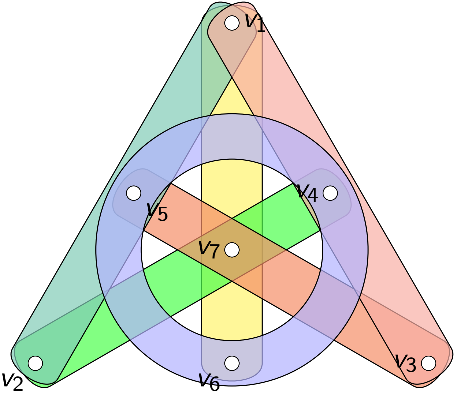

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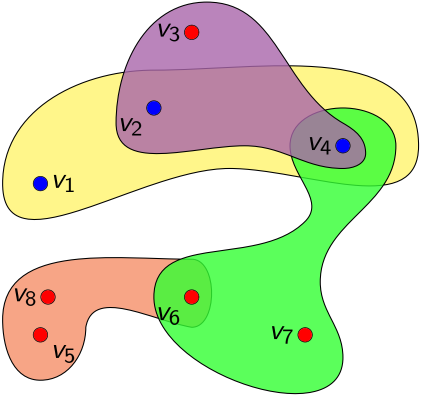

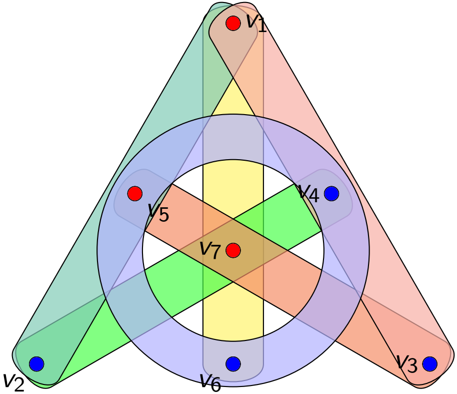

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no monochromatic edges

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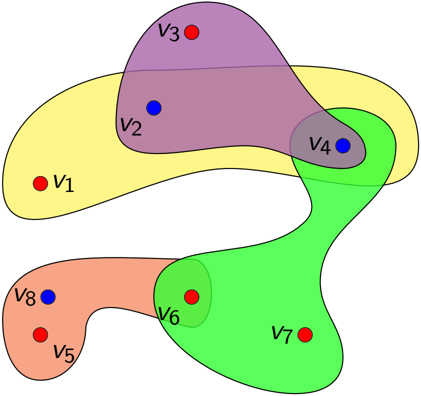

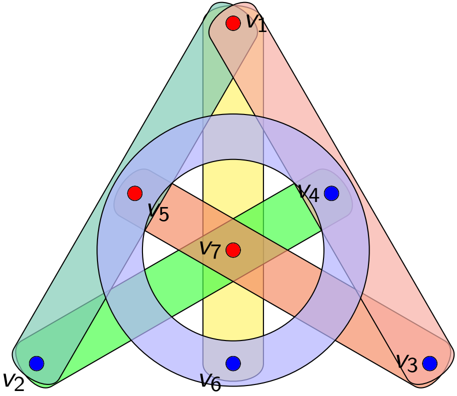

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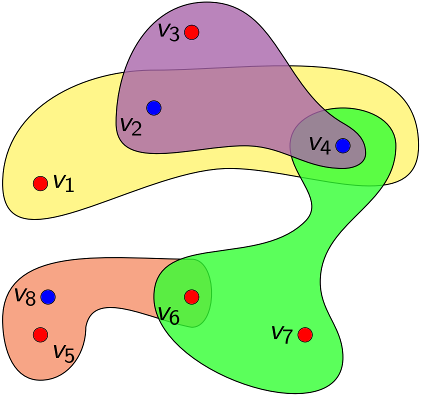

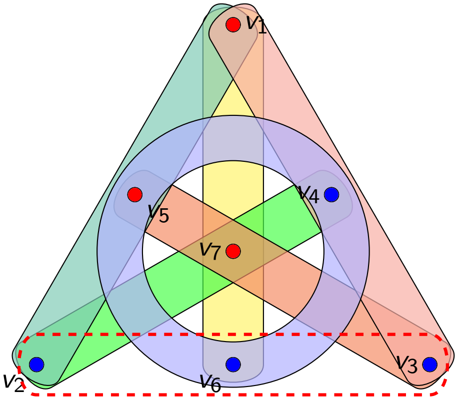

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$m(k)$ - the minimum number of edges in a k-graph that is not two colorable.

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Observation

Lower bound, can be improved for (pseudo-)random k-graphs.

Sparse/Dense constraints

Lovász Local Lemma

Any k-graph with the maximum edge degree at most

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(Erdős' upper bound is built on $k^2/2$ vertices.)

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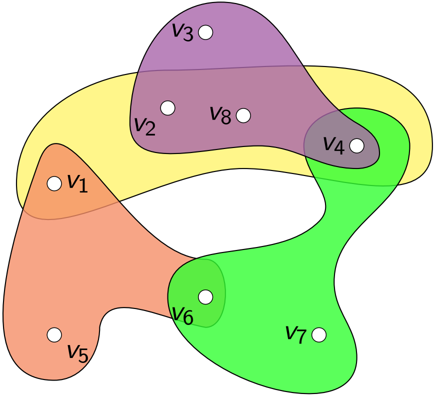

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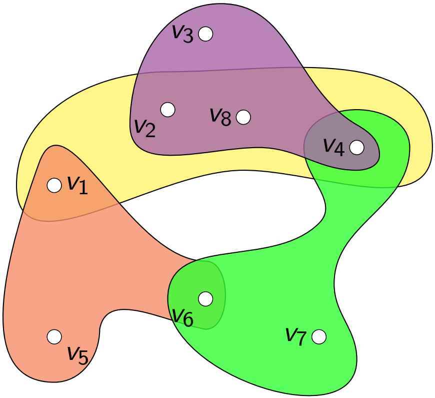

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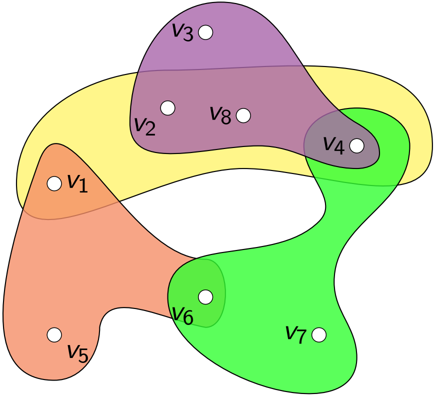

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<u>k^+-graphs</u>

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Does there exist an unbounded function $f(k)$ such that any k^+-graph H that satisfies

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Yes! Take $f(k) \approx \log^{*}(k)$.

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Duraj, Gutowski, JK 2018

$f(k) \approx \log(k)$ works as well.

(k, k')-graphs

$k < k'$

H is a $\underline{(k, k')\text{-graph}}$ if every edge of H is of size either k or k'.

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the weakest link of the proof

For every long edge f we define r.v. X_f. We manage to impose constraints

$$\mathbb{E}[X_f] \leqslant 2q \quad \text{and} \quad X_f \leqslant qk.$$

Then we work with the worst-case distribution satisfying above conditions.

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We can improve lower bounds for H in which X_H is concentrated.

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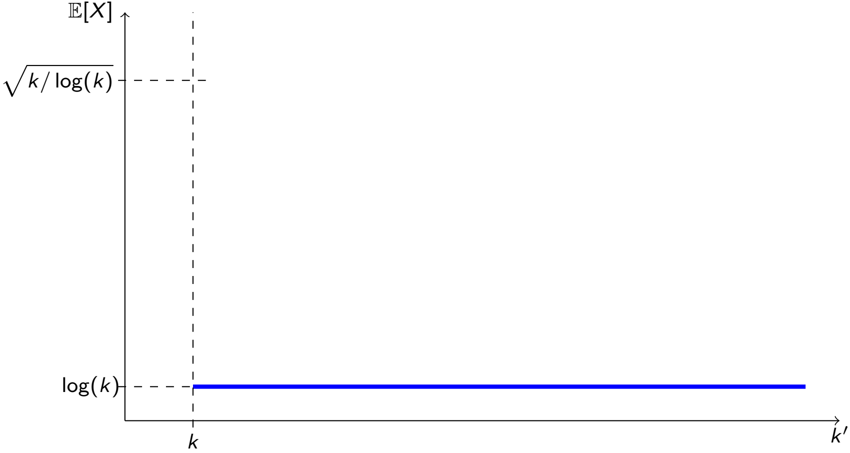
In the uniform case ...

For k-graphs we have stronger constraint $X_f \leqslant k$.

(k, k')-graphs: what is the worst k'?



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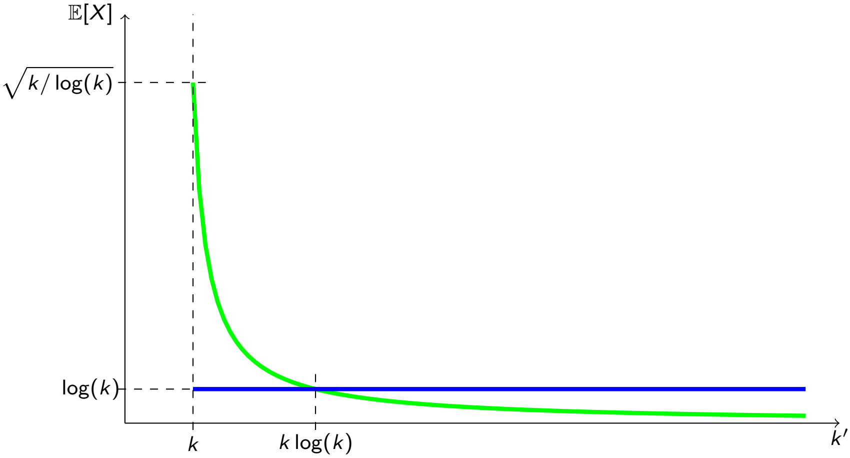

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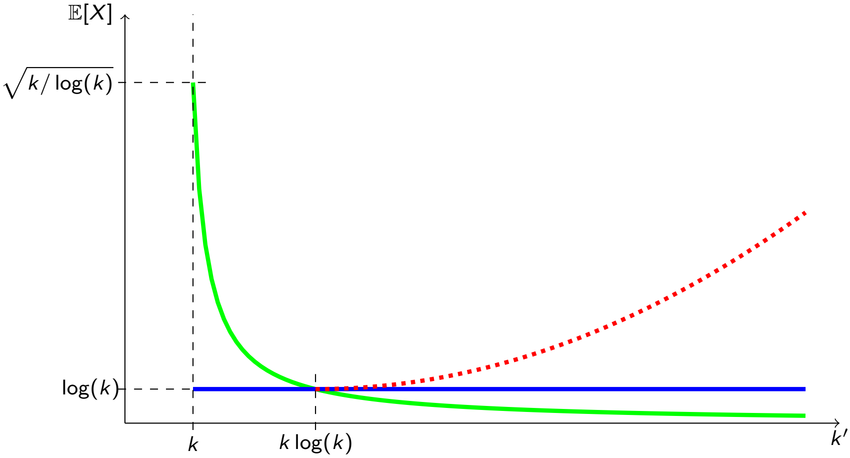

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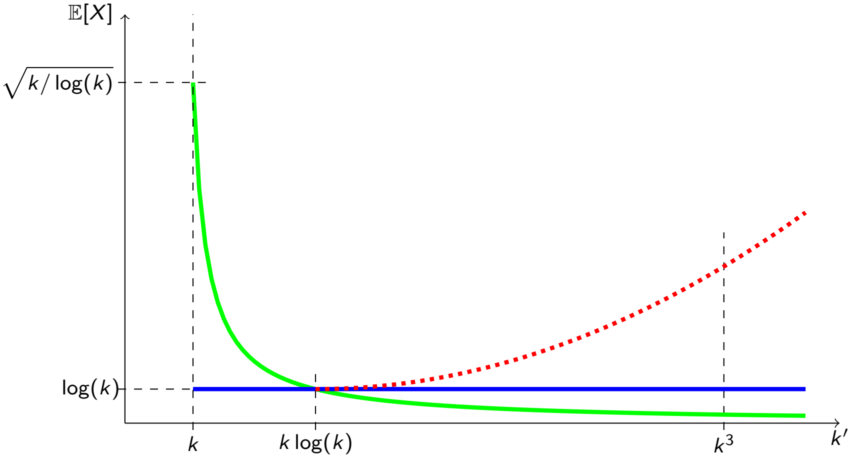
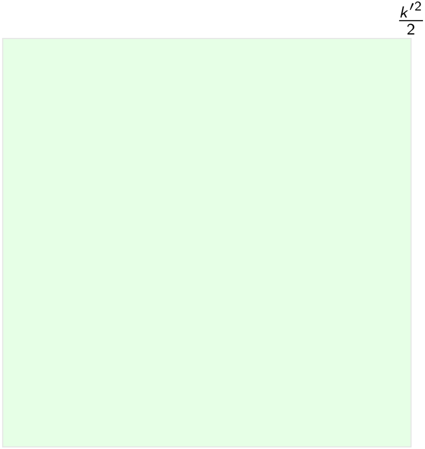


Landscape of the worst instance of (k, k')-graphs



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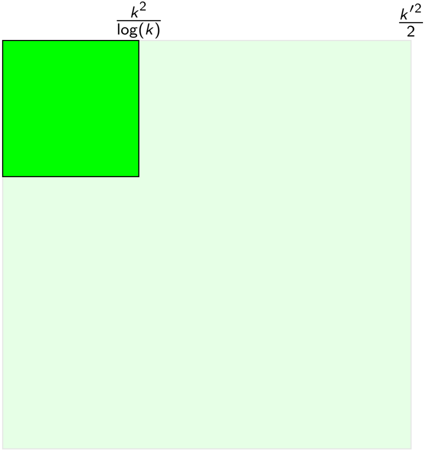

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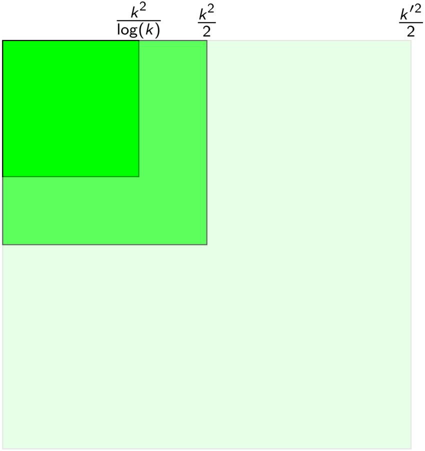

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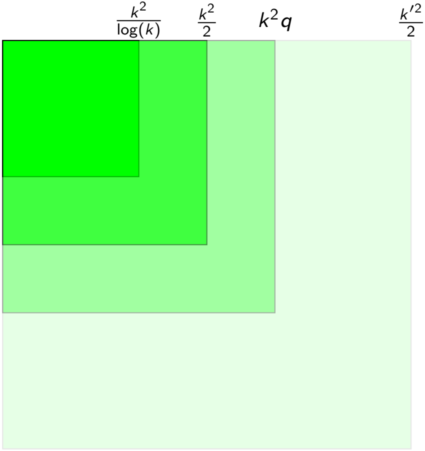

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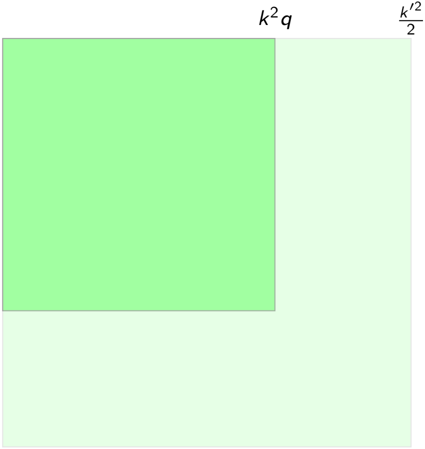

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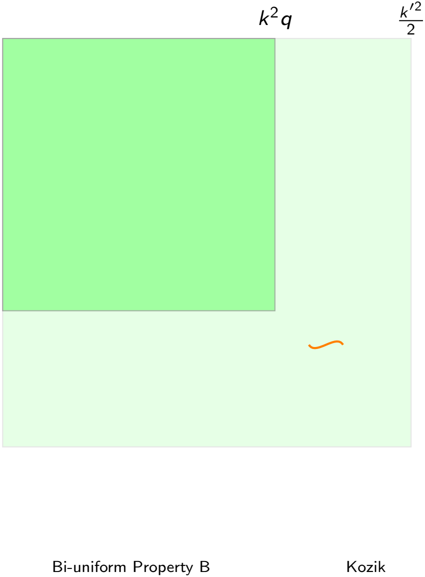

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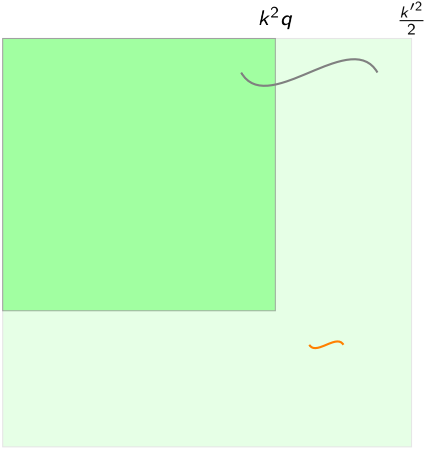

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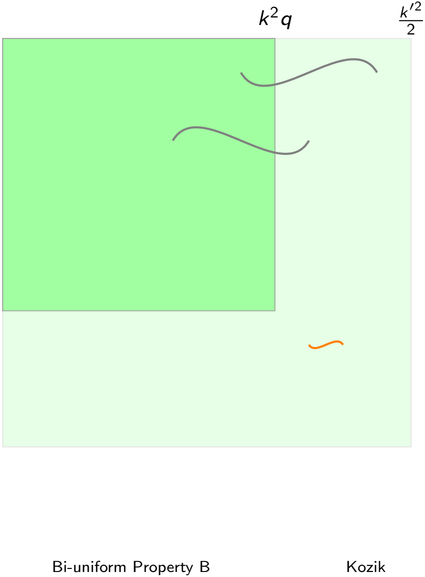

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