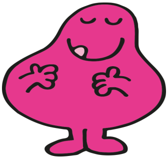
Greedy maximal independent sets via local limits

Peleg Michaeli

Tel Aviv University

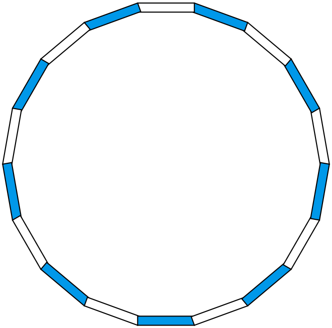
Probabilistic Combinatorics Online, 25 September 2020



Joint work with Michael Krivelevich, Tamás Mészáros and Clara Shikhelman

31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods
for the Analysis of Algorithms (AofA 2020)

Parking cars on a cycle



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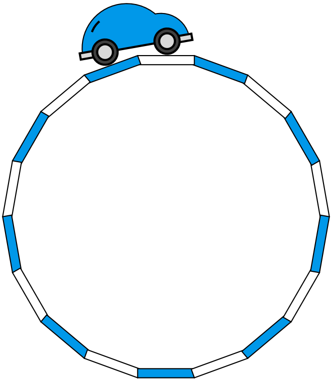

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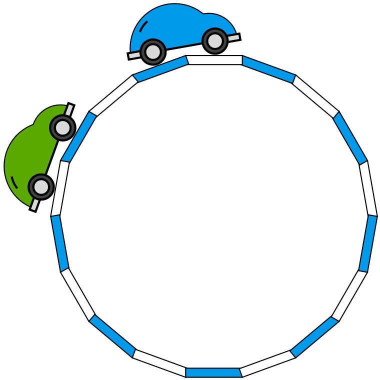

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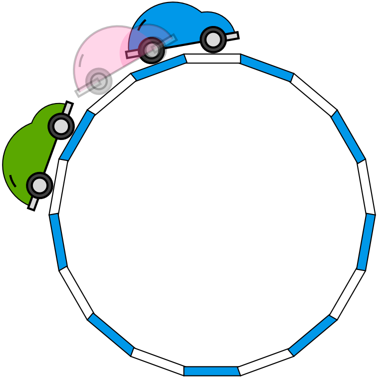

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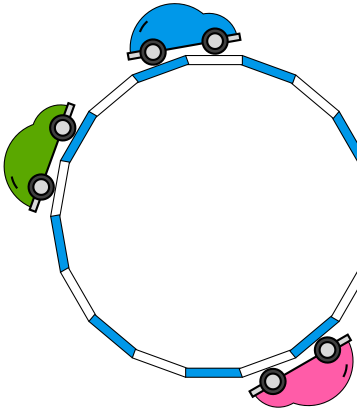

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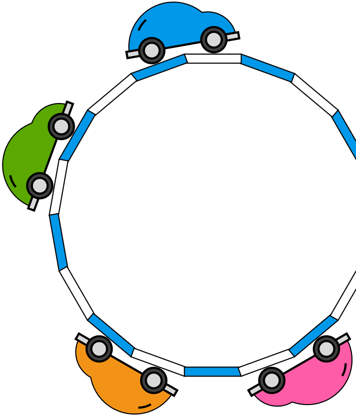

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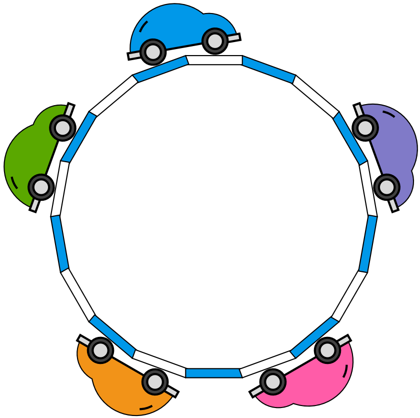

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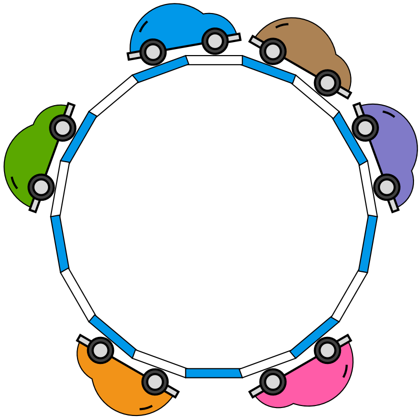

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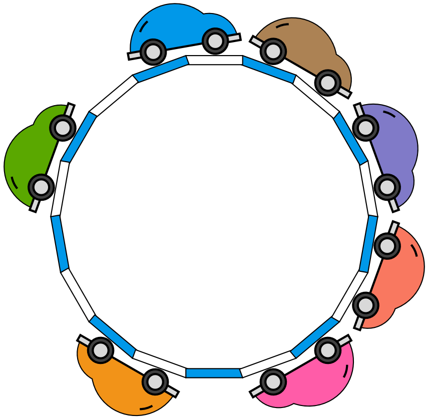

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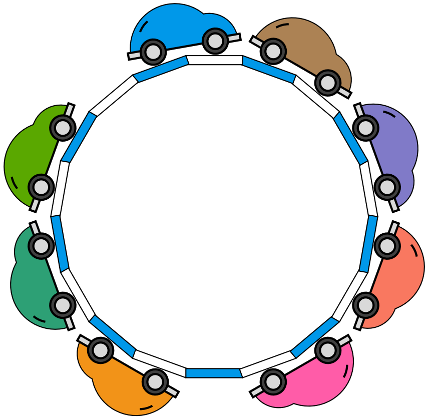

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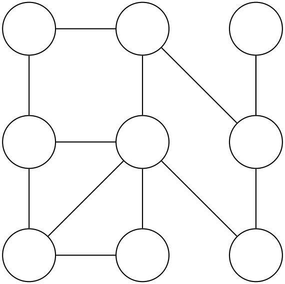
Greedy MIS



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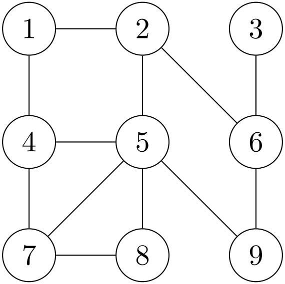

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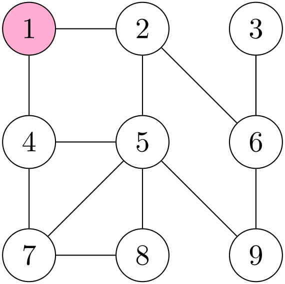

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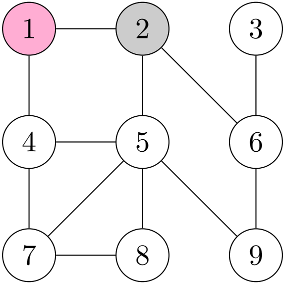

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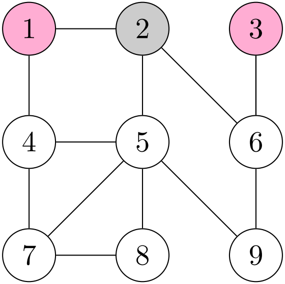

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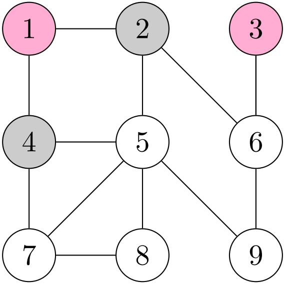

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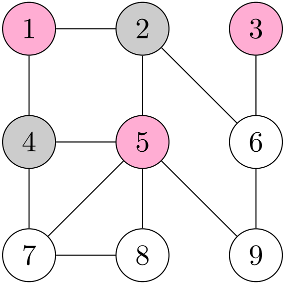

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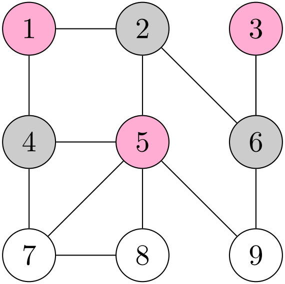

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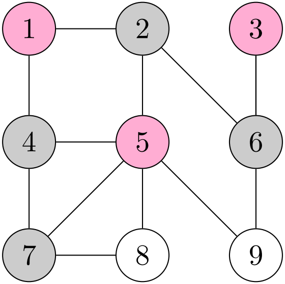

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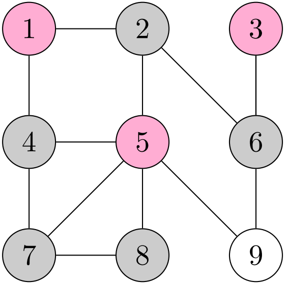

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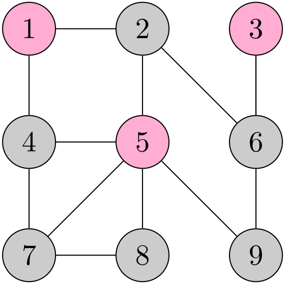
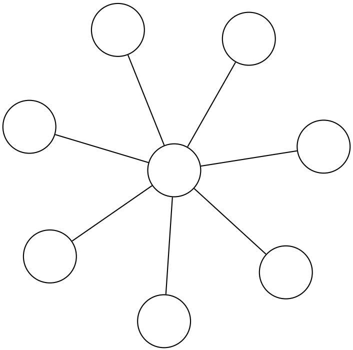


Greedy MIS — performance



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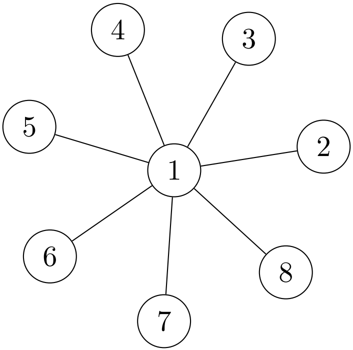

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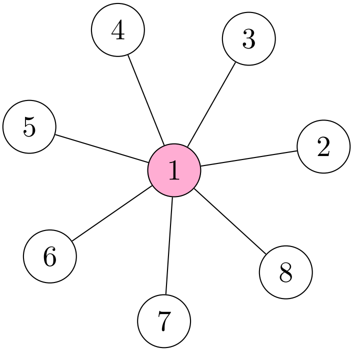

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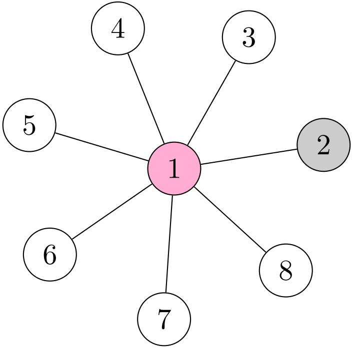

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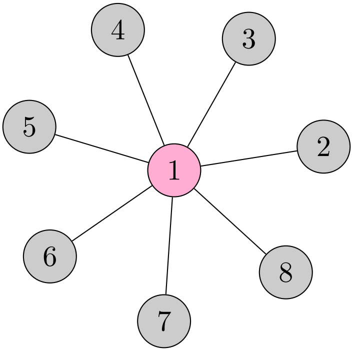

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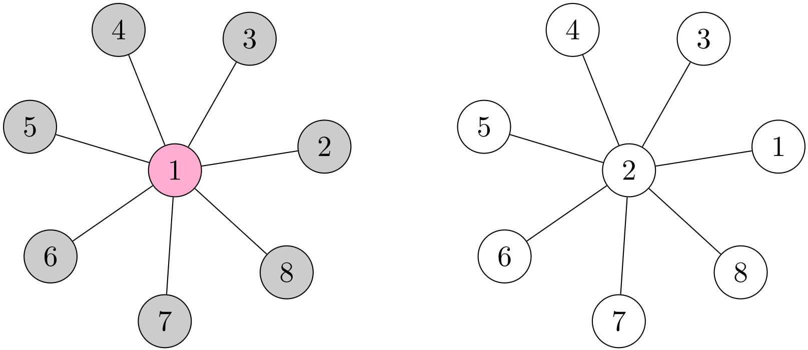

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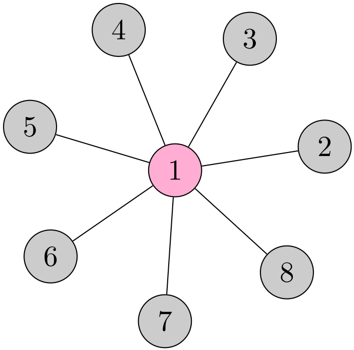

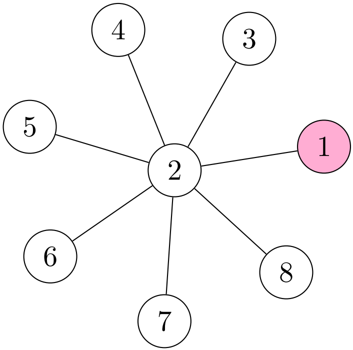

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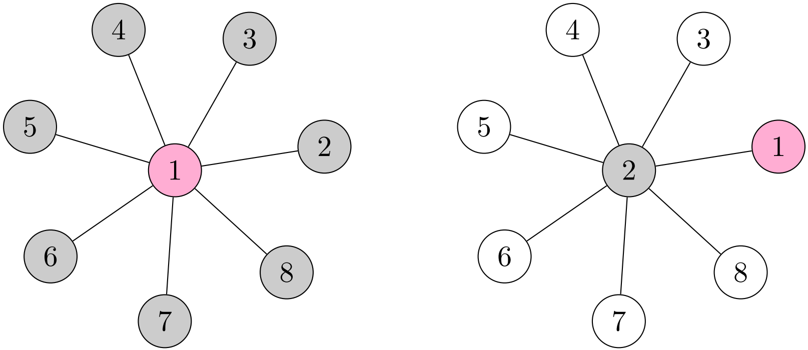

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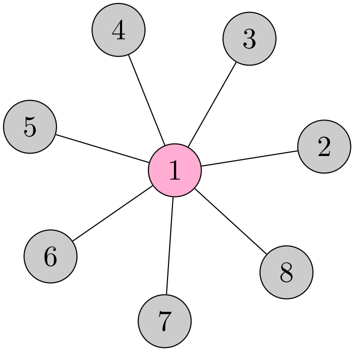

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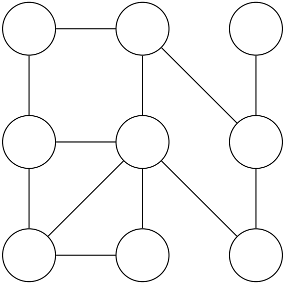

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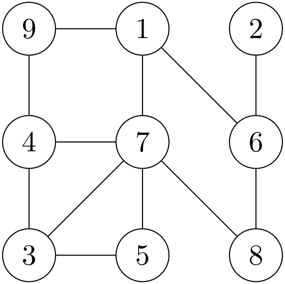

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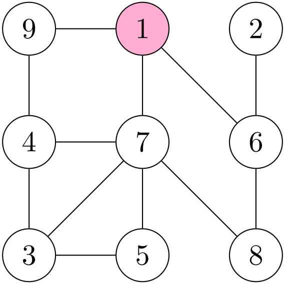

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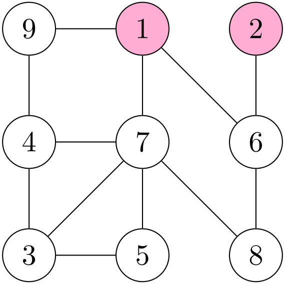

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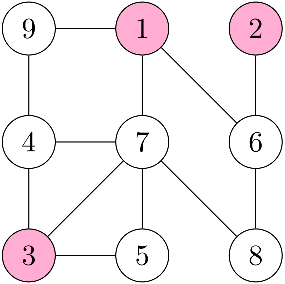

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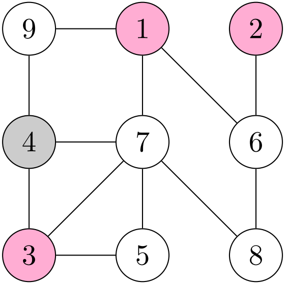

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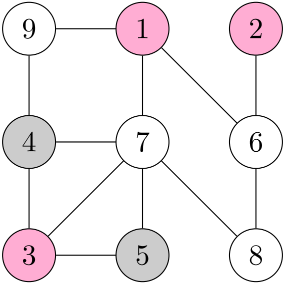

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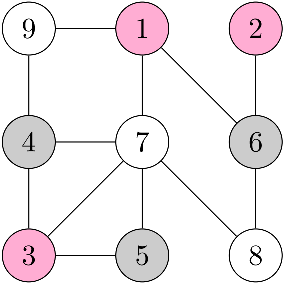

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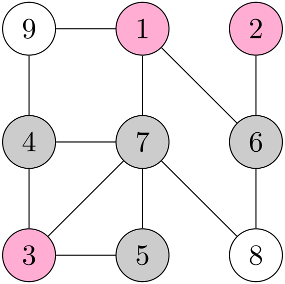

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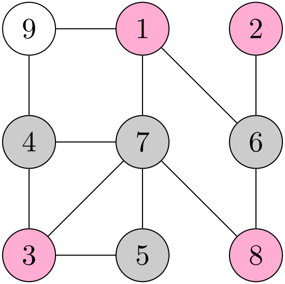

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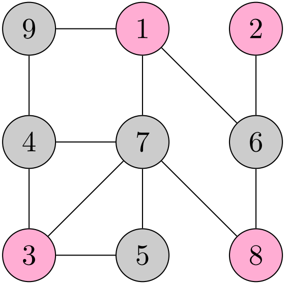

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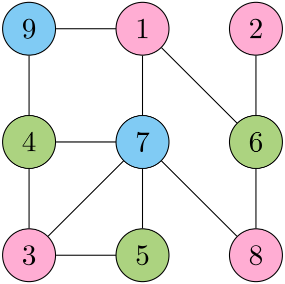

Greedy independence ratio — previous work

Let $\mathbf{I}(G)$ be the yielded independent set, and let $\iota(G) = |\mathbf{I}(G)|/|V(G)|$.

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BJL '17, BJM '17	random graphs with given degree sequence

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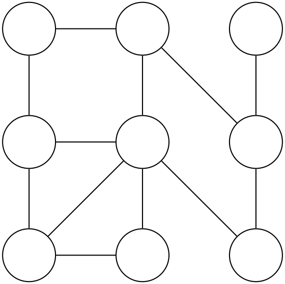

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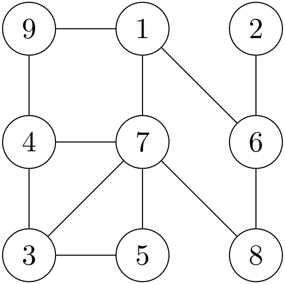

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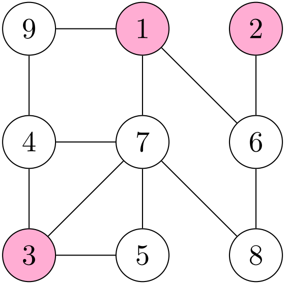

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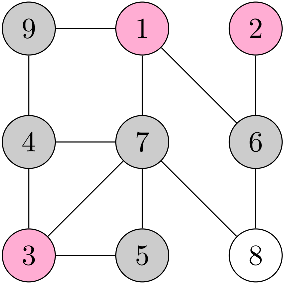

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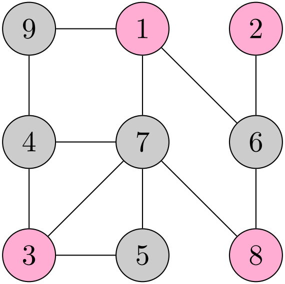

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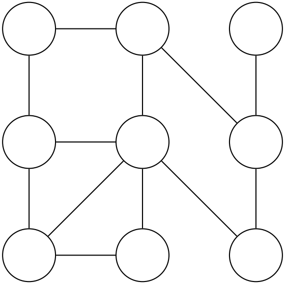

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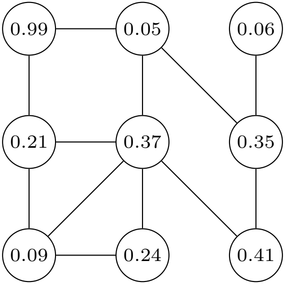

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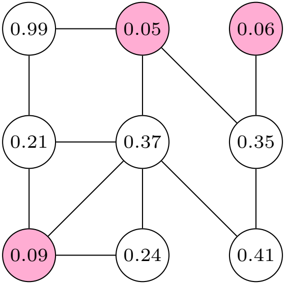

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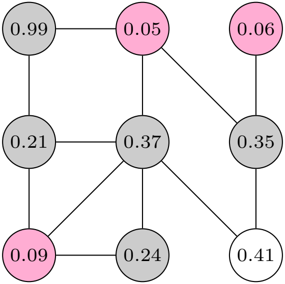

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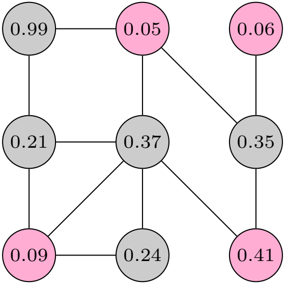

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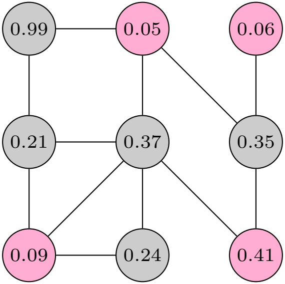

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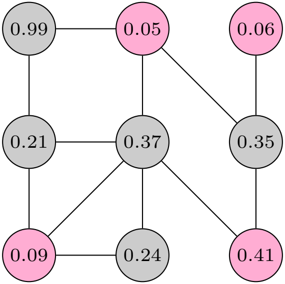

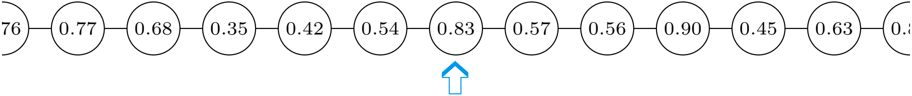

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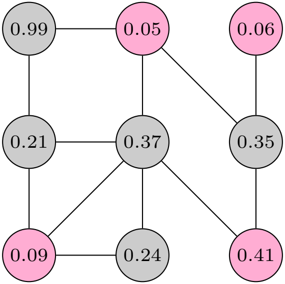

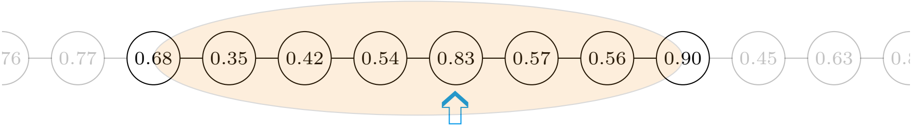

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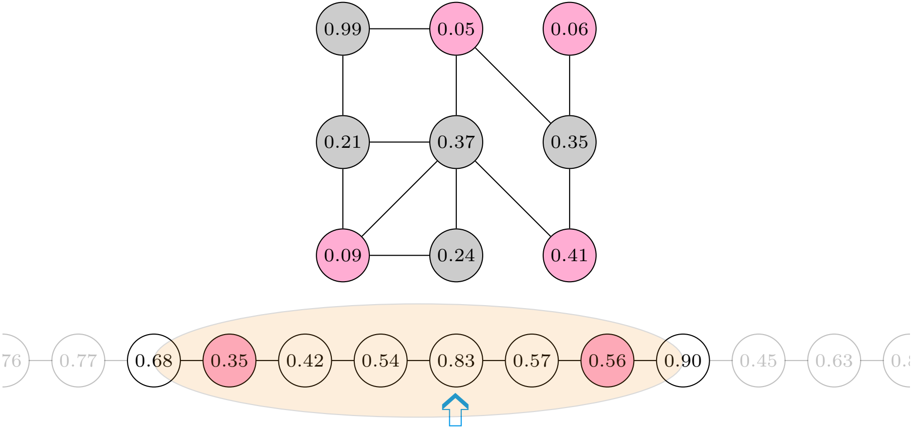

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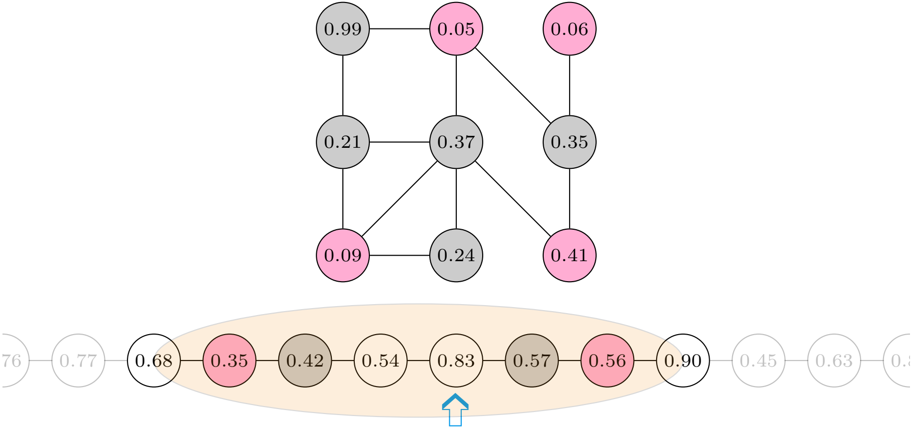

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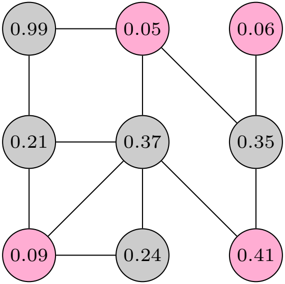

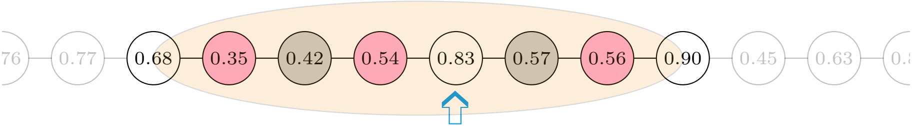

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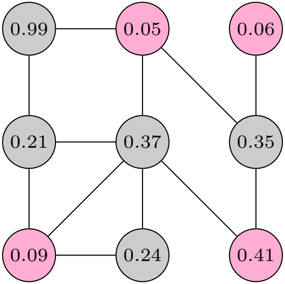

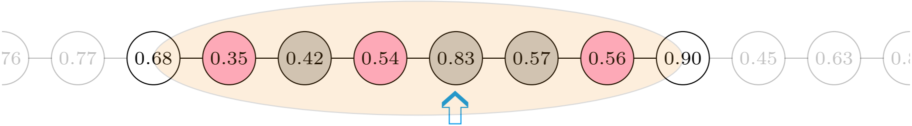

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Local limits (a.k.a. Benjamini–Schramm Limits)

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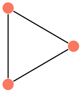

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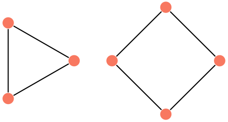

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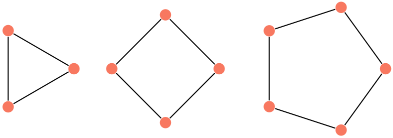

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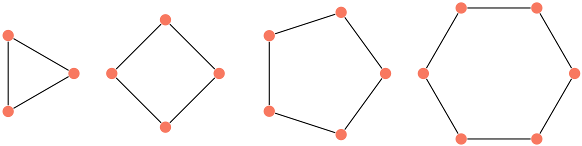

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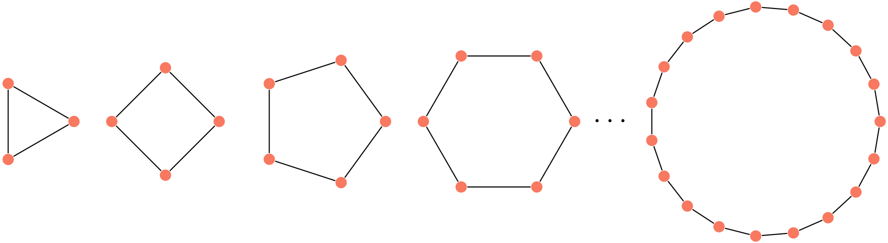

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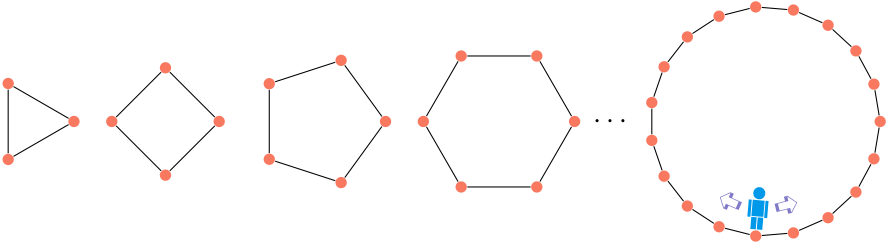

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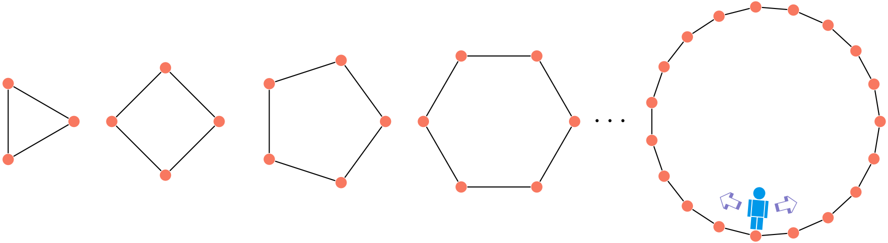

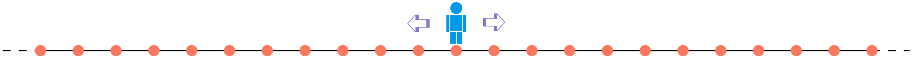

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Examples

- $P_n, C_n \xrightarrow{\text{loc}} \mathbb{Z}$
- $[n]^d \xrightarrow{\text{loc}} \mathbb{Z}^d$
- $G(n, d/n) \xrightarrow{\text{loc}} \mathcal{T}_d$, a Galton–Watson Pois$(d)$ tree
- $G_{n,d} \xrightarrow{\text{loc}}$ the d-regular tree
- Uniform random tree $T_n \xrightarrow{\text{loc}} \hat{\mathcal{T}}_1$, a size-biased GW Pois$(1)$ tree
- Finite d-ary balanced tree $\xrightarrow{\text{loc}}$ the canopy tree

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Exploration algorithms / decay of correlation

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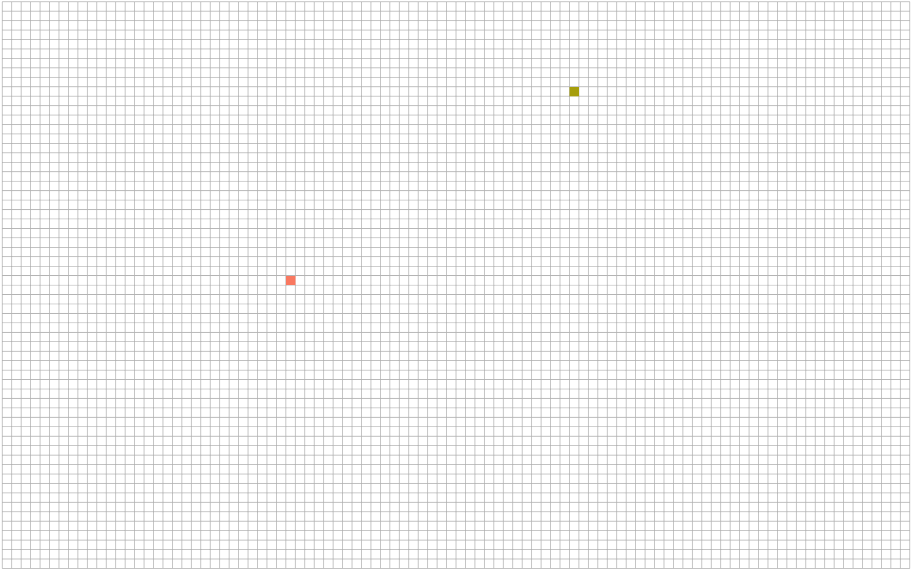

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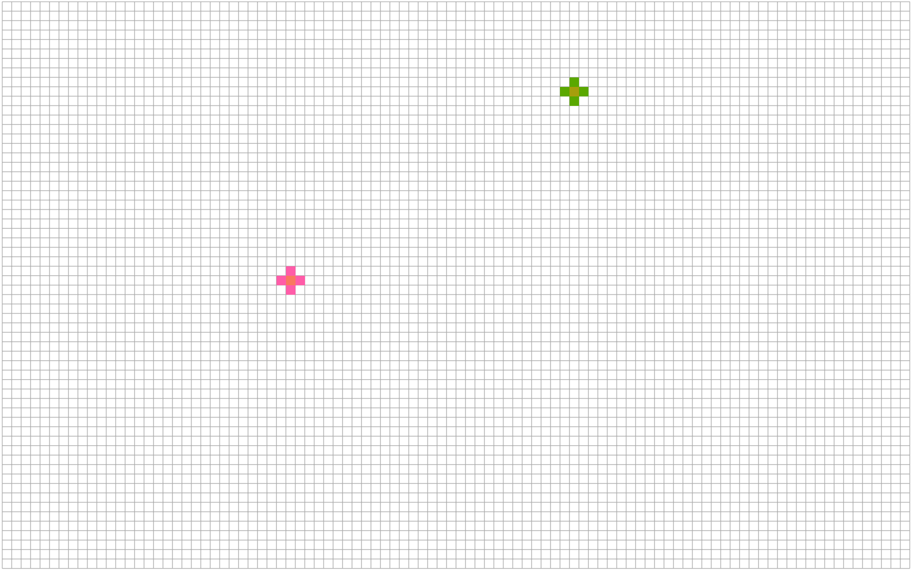

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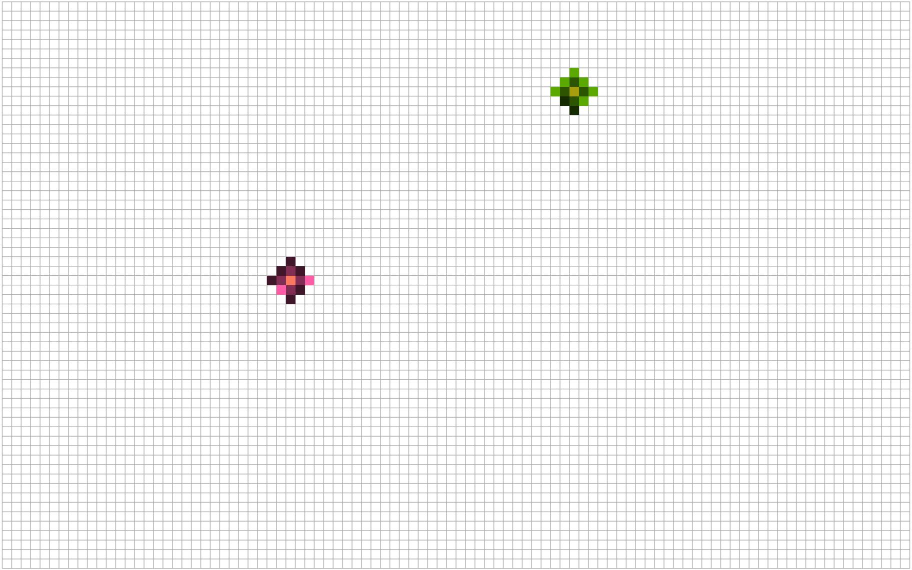

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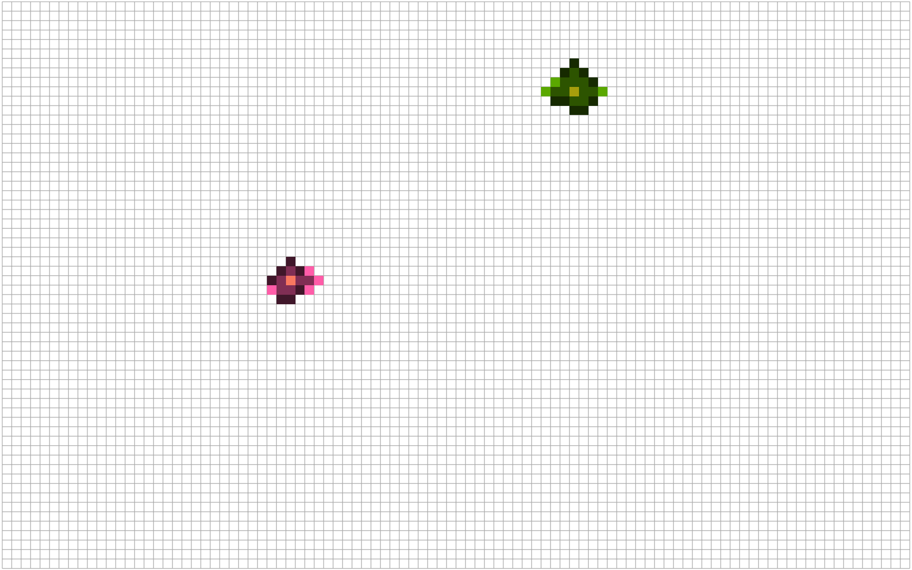

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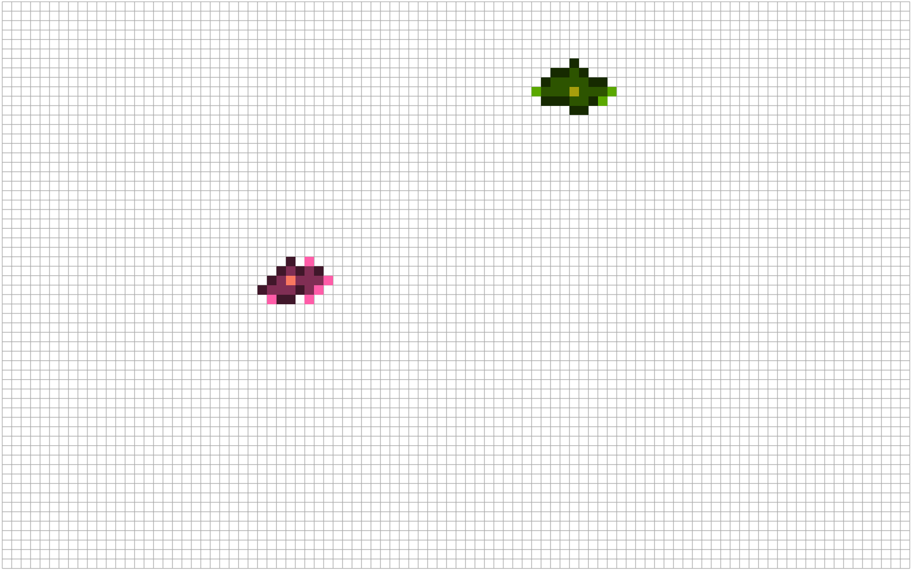

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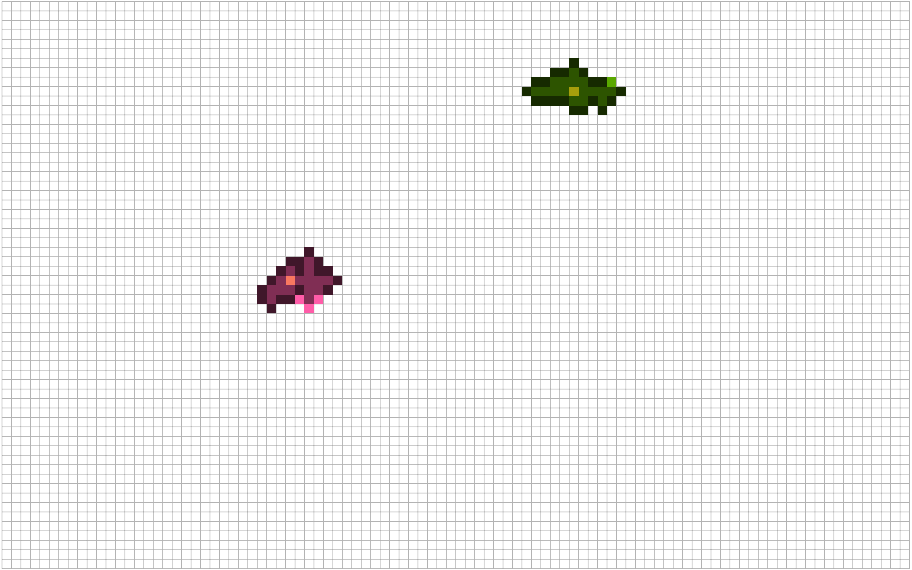

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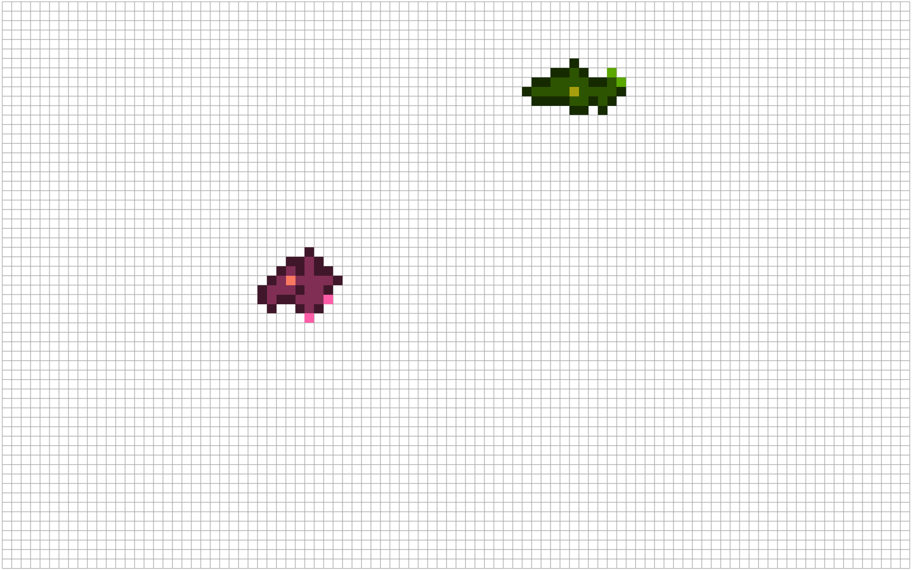

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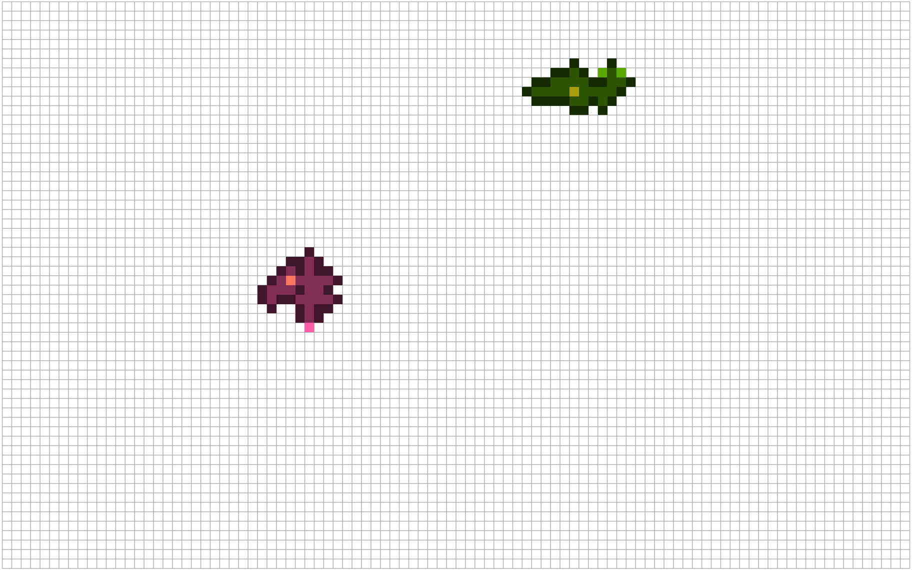

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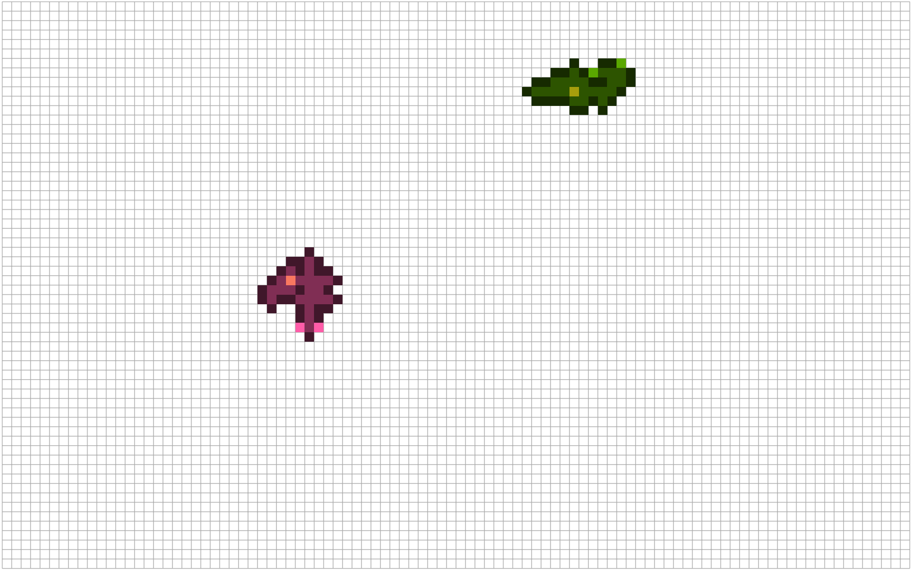

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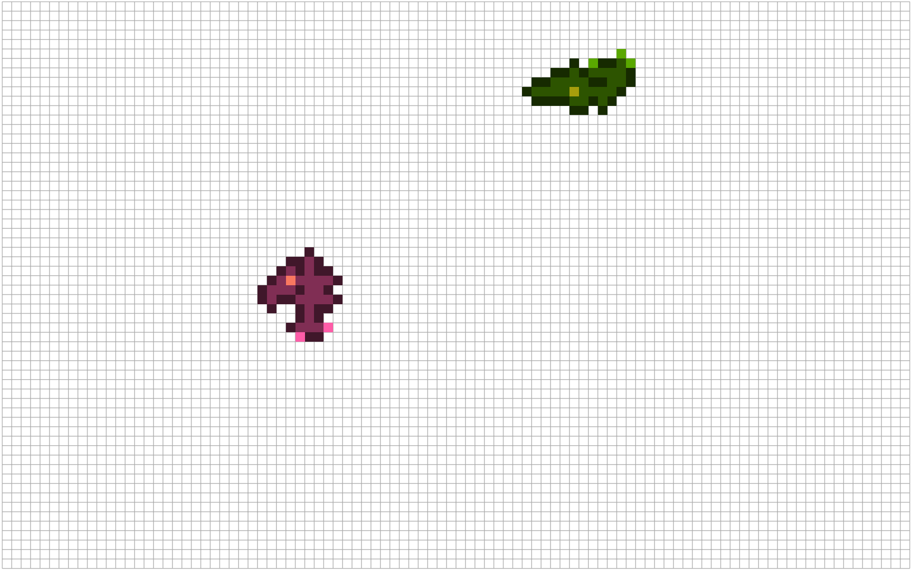

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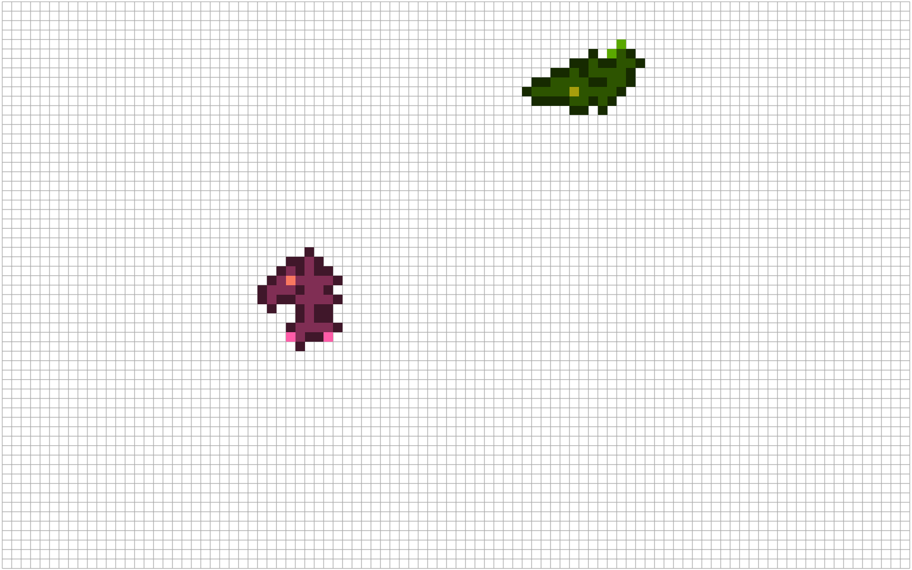

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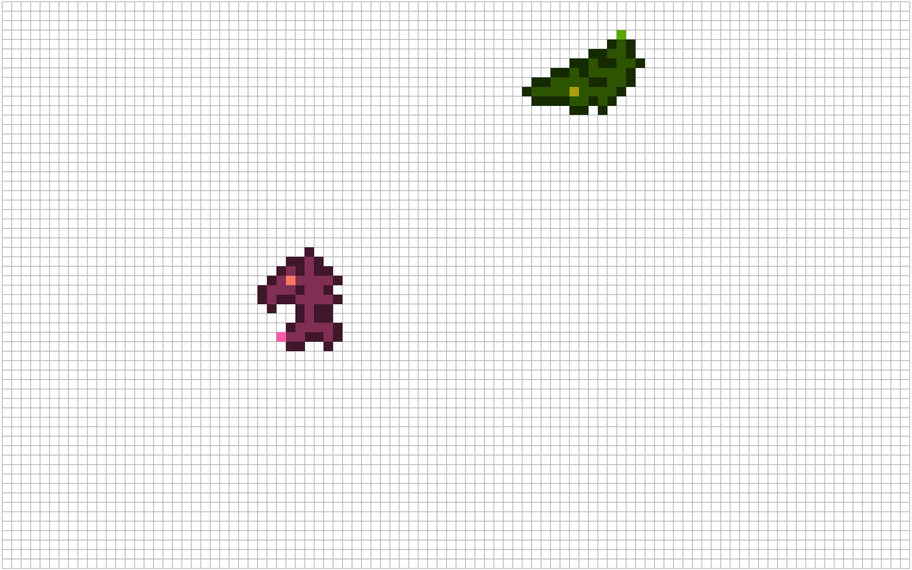

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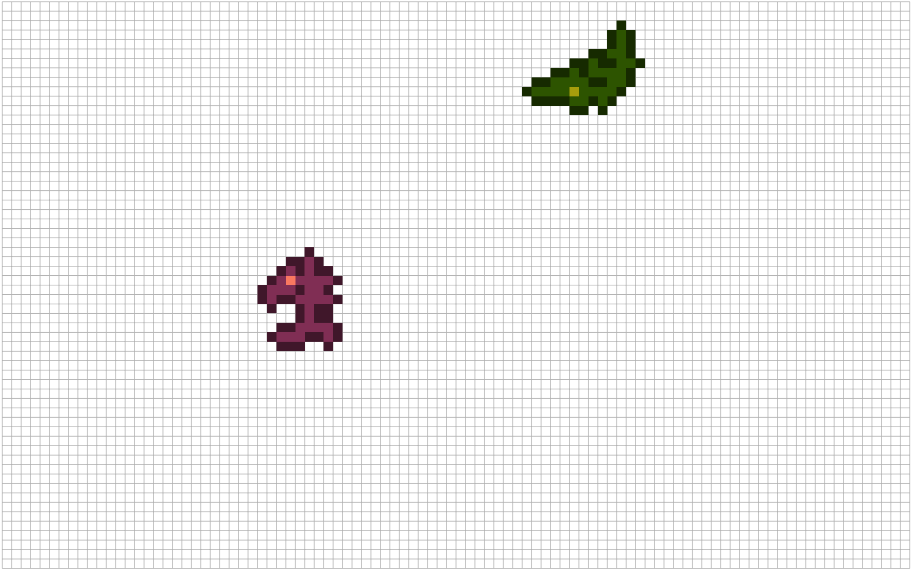

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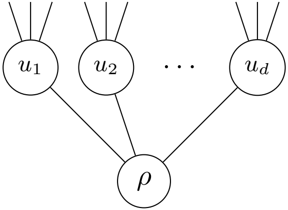

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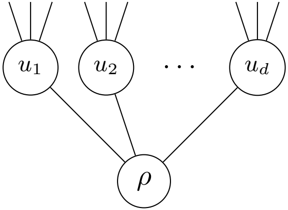

Assuming the children of ρ are roots to independent subtrees, and conditioning on the label of ρ, children of the *past* are roots to independent processes.

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Thus, if y is a unique solution of

$$y'(x) = \sum_{\ell \in \mathbb{N}} \mathbb{P}(\xi[< x] = \ell) \left(1 - \frac{y(x)}{x}\right)^\ell, \qquad y(0) = 0,$$

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Systems of ordinary differential equations

Let (U, ρ) be a (simple) multitype branching process.

$$\begin{aligned} y_k(x) &= \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \wedge \sigma_\rho < x \mid \tau = k) \\ &= x \cdot \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \mid \sigma_\rho < x, \tau = k) \\ &= \int_0^x \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \mid \sigma_\rho = z, \tau = k) dz \\ y_k'(x) &= \mathbb{P}(\rho \in \mathbf{I}(U[\mathcal{P}_\rho]) \mid \sigma_\rho = x, \tau = k) \end{aligned}$$

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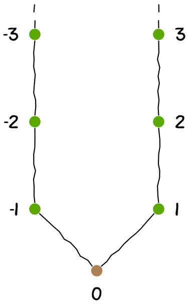

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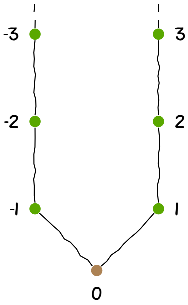

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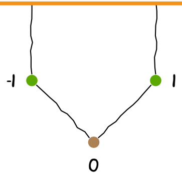

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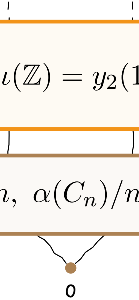

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Easy fact: $G(n, d/n)$ converges locally to the Pois(d) branching process.

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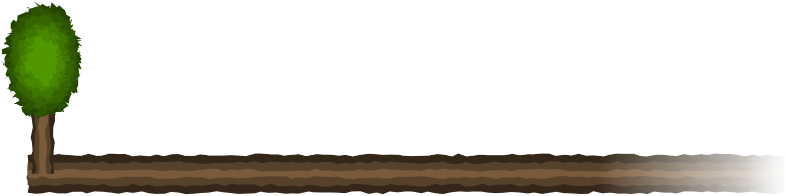

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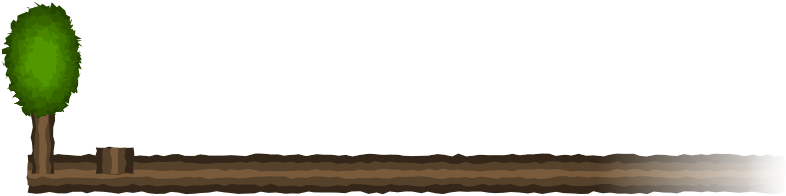

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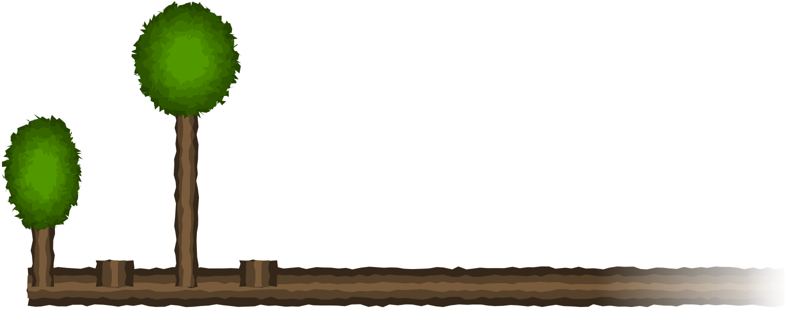

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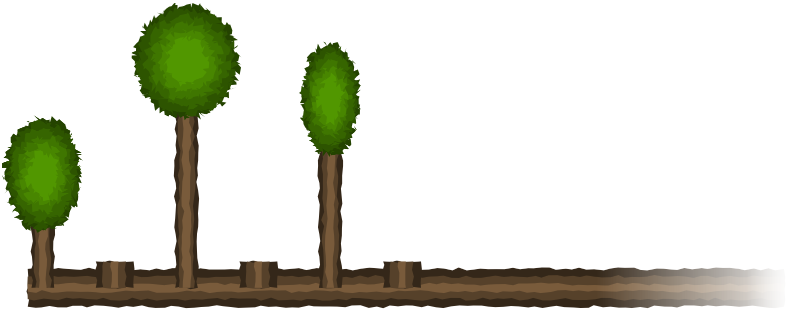

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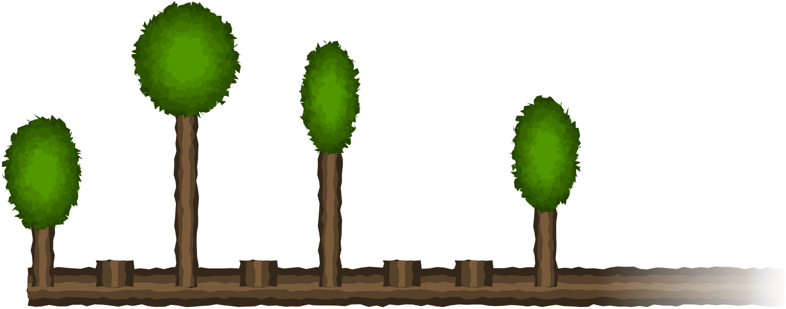

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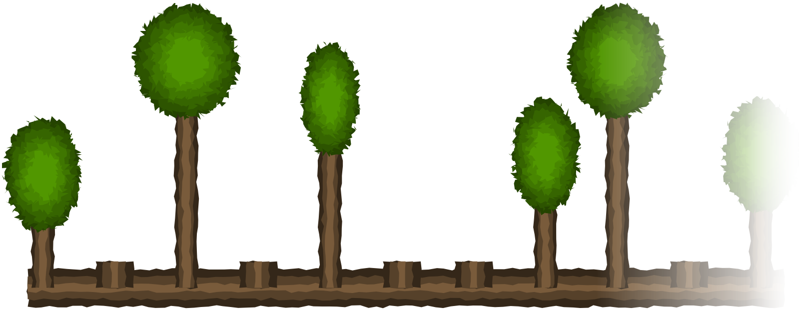

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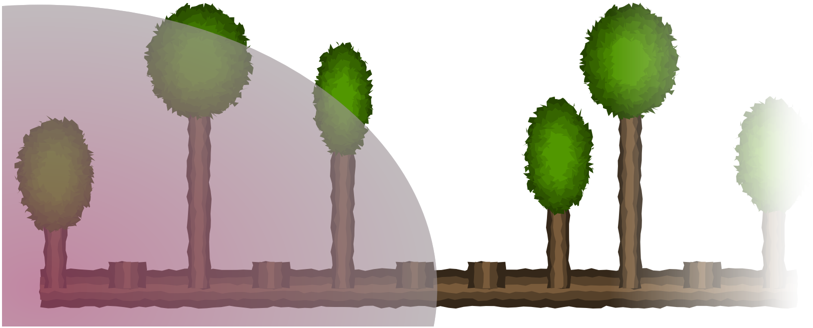

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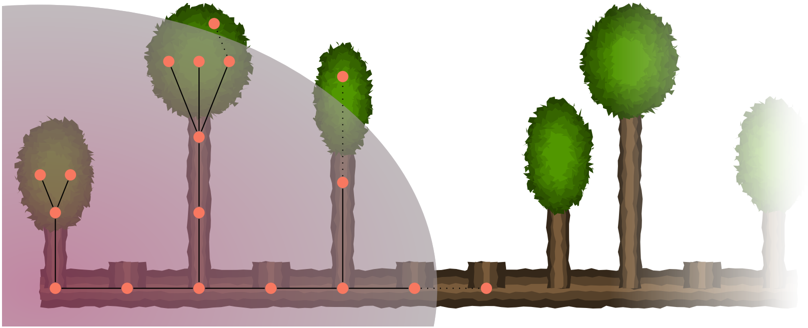

Application: uniform random trees

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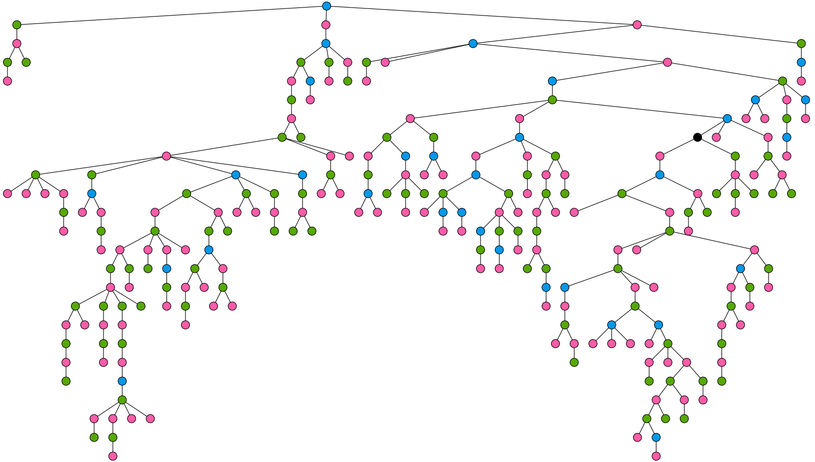

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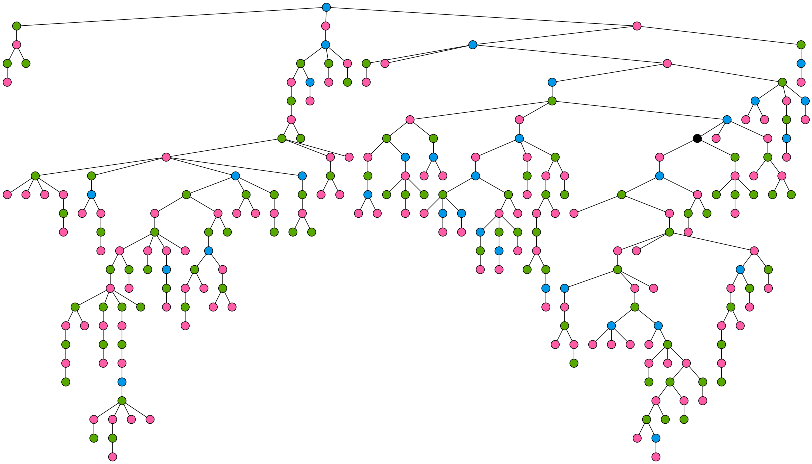
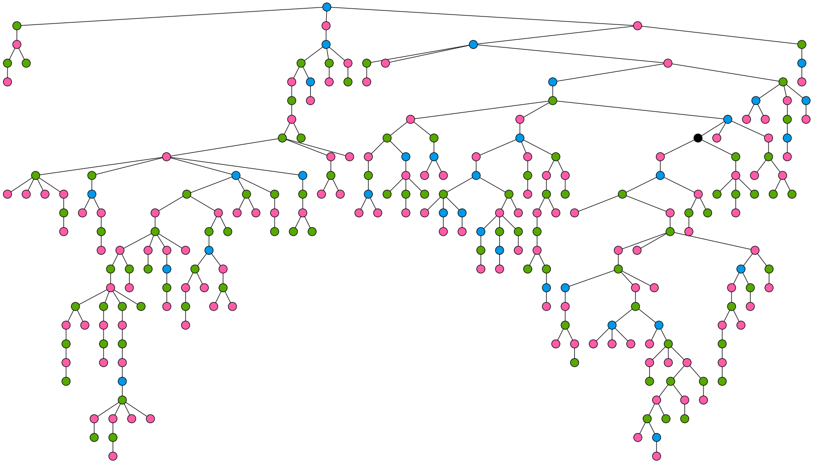


red: 125 (50%), green: 92 ($\approx$ 37%), blue: 32 ($\approx$ 13%), black: 1

Simulations don't lie (but I do)



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Flory '39, Page '59 $\iota(P_n) \to \frac{1}{2}(1 - e^{-2})$

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	(same for functional digraphs)	✸

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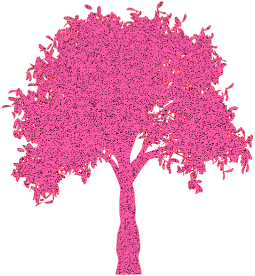

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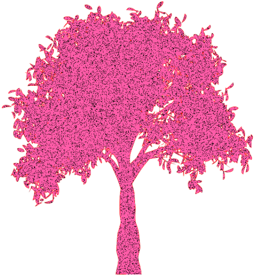
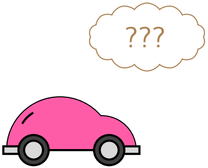


Theorem (Krivelevich, Mészáros, M., Shikhelman '20)

If T is a tree on n vertices, then $\mathbb{E}[\iota(P_n)] \leq \mathbb{E}[\iota(T)]$.

What's next?

- Graph sequences that are not locally tree-like
- Better/other local rules
- Other colours



Thank You!