The chromatic number of a random lift of a regular graph

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joint work with

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Lifts

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Surjective graph homomorphism $\Pi : L \to G$ that is a bijection between edges incident with v and edges incident with $\Pi(v)$.

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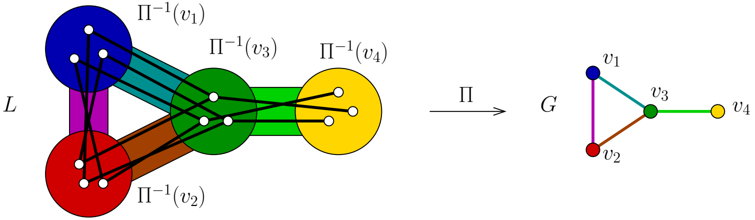

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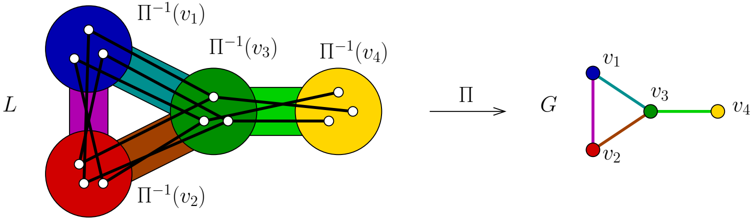
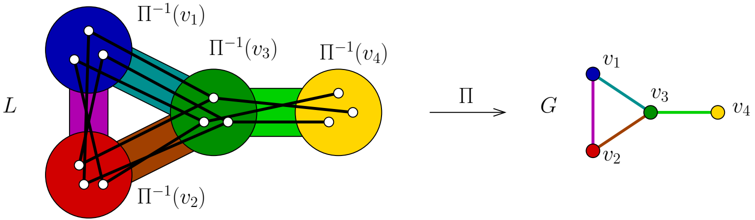


Note: G may have loops and multiple edges.



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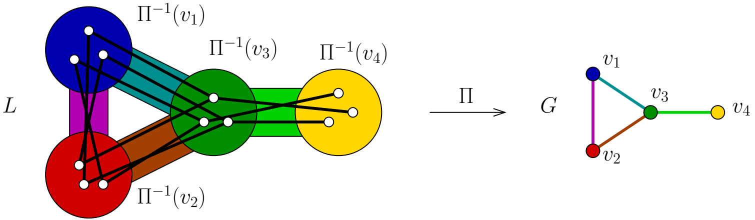
L is an ***n*-lift** if all $|\Pi^{-1}(v)| = n$.



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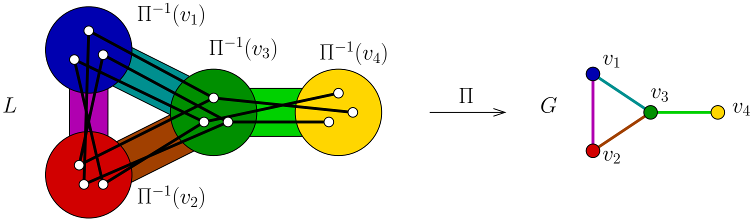
Fact 2: G d-regular $\Longrightarrow$ any lift L is d-regular.
(loops contribute 2 to degree.)

Properties of lifts



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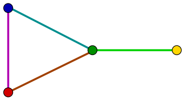
Fact 3: $\chi(L) \leq \chi(G)$

Random n-lifts

G



Random n-lift model (Amit, Linial 2002):

- Replace $v \in V(G)$ by bin with n vertices.
- Replace $e \in E(G)$ by random perfect matching.
  (slightly different for loops.)

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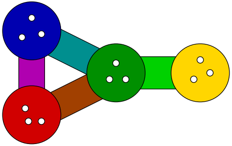

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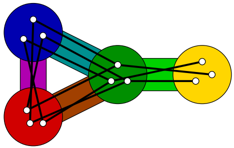

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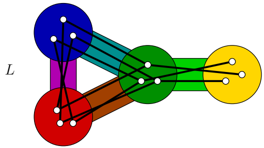

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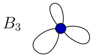

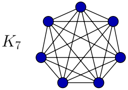

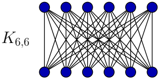

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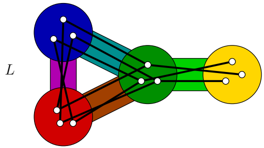

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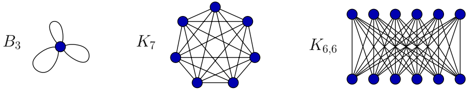

Facts (for d-regular G):

- $G = B_{d/2} \implies L$ contiguous to uniform d-regular multigraph.
- Not true for $G = K_{d+1}$

(Amit, Linial, Matoušek 2002)

Problem 1

For any G: is $\chi(L) = \Omega\left(\frac{\chi(G)}{\log \chi(G)}\right)$ a.a.s.?

Known for $G = K_{d+1}$.

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(Known that a.a.s. $\chi(L) \in \{3, 4\}$.)

(Still) open questions

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Conjecture

For any G, there is k_G with $\chi(L) = k_G$ a.a.s.

. . . and many more open questions!

Our results

Thm (P-G, Nir 2019++):

Let $d \geq 3$ and $k_d = \min\{k \in \mathbb{N} : d < 2k \log k\}$. $(k_d \approx \frac{d}{2 \log d})$

Let L be random n-lift of $G = K_{d+1}$.

Then a.a.s. $\chi(L) \in \{k_d, k_d + 1\}$.

For half of the d, a.a.s. $\chi(L) = k_d$.

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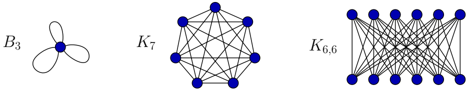

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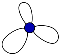
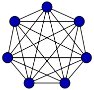
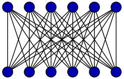
B_3 

K_7 

$K_{6,6}$ 

Thm (Kemkes, P-G, Wormald 2010):

Analogous result holds for uniform d-regular graphs.

(Improved by Coja-Oghlan, Efthymiou, Hetterich 2016.)

Main tools

- Small subgraph conditioning method
  (Robinson & Wormald 1992)
- Optimization over stochastic matrices
  (Achlioptas, Naor 2005)
- Laplace summation method
  (Greenhill, Janson, Ruciński 2010)
- Saddle-point method
- Algebraic graph theory
  - Kirchhoff Matrix-Tree Thm
  - Counting non-backtracking closed walks

Lower bound on $\chi(\boldsymbol{L})$:

$X = \#$ k-colourings of L.

Thm: If $k < k_d$, then $\mathrm{E}X = o(1)$

Then $\mathrm{P}(X > 0) \leq \mathrm{E}X = o(1)$.

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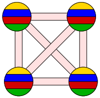

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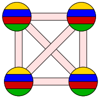

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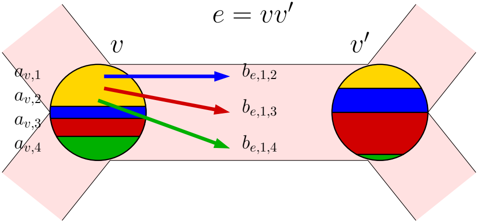
Then $\mathrm{P}(Y > 0) \geq \frac{(\mathrm{E}Y)^2}{\mathrm{E}Y^2} \sim C > 0$ (Paley-Zygmund)

Unfortunately, $C < 1$ due to the influence of short cycles in L.

Estimating moments I

$$\mathrm{E}X = \sum_{\boldsymbol{a},\boldsymbol{b}} T(\boldsymbol{a}, \boldsymbol{b}, n)$$



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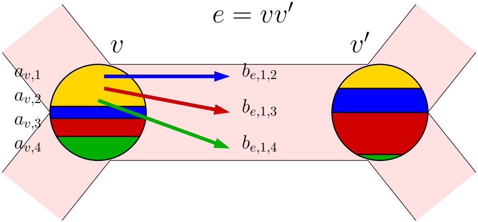

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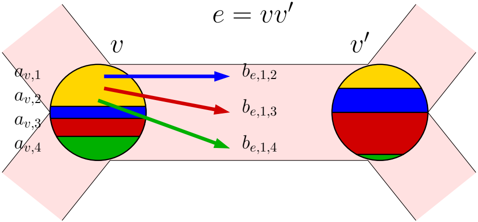

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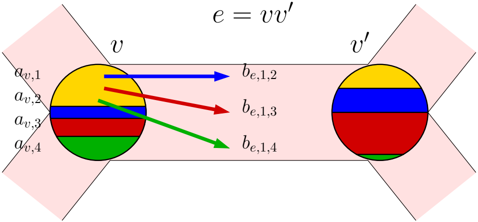

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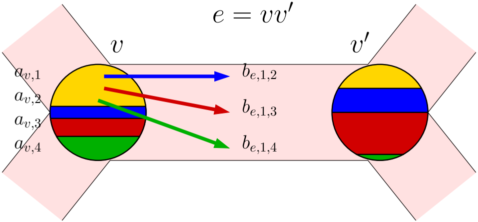

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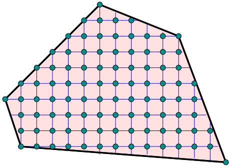
Claim:

Max contribution is from $a_{v,i} = 1/k$, $b_{e,i,i'} = 1/k(k-1)$.

(We extend result by Achlioptas, Naor 2005.)

Estimating moments II

Summation domain:



$$\begin{cases} \boldsymbol{x} \in \mathcal{P} \\ B\boldsymbol{x} = \boldsymbol{y} \\ \boldsymbol{x} \in \left(\frac{1}{n}\mathbb{Z}\right)^D \end{cases}$$

where:

- $\mathcal{P} \subset \mathbb{R}^D$ polytope
- $\dim(\operatorname{Ker} B) = r$

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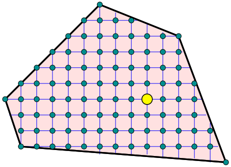

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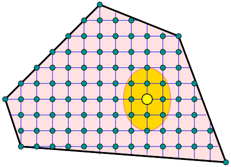

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  (Laplace summ. / Saddlepoint method)

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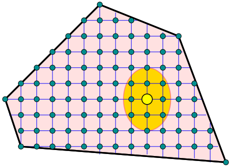

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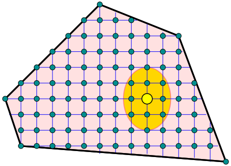

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- About C: It depends on

  $$\begin{cases} \text{Hessian of } f \\ \text{Volume of fundamental cell in lattice} \end{cases}$$

  (Greenhill, Janson, Ruciński 2010)

  Instead, we count maximal forests in Γ with incidence matrix B.

Small subgraph conditioning (Robinson, Wormald 1992)

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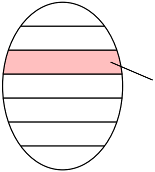
$Z_i = \#$ cycles of length i in L.

$Z_i \sim \mathrm{Poi}(\lambda_i)$ (+ asymptotic independence)



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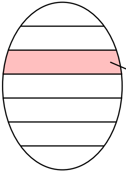

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Rough idea:

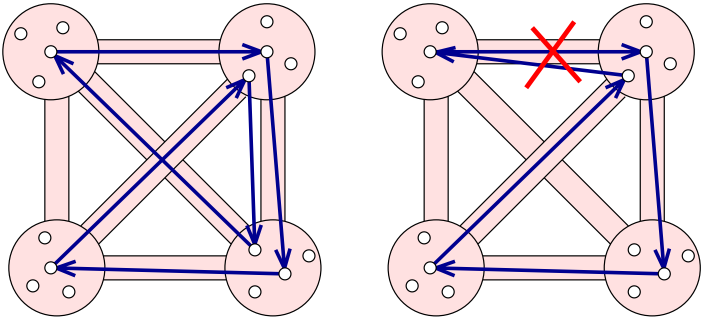
Suppose:

- $$\frac{\mathrm{E}(YZ_i)}{\mathrm{E}Y} = 1 + \delta_i + o(1)$$ (& joint factorial moments)

  (i.e. $Z_i \sim \mathrm{Poi}(1+\delta_i)$ in space "weighted" by Y).

- $$\frac{\mathrm{E}Y^2}{(\mathrm{E}Y)^2} = \exp(\textstyle\sum_i \lambda_i \delta_i^2) + o(1).$$

Then $\mathrm{P}(Y > 0) \to 1$ (+ contiguity [...]).



Some algebraic tools: (Friedman 2008)

(Still) open questions

(Amit, Linial, Matoušek 2002)

Problem 1

For any G: is $\chi(L) = \Omega\left(\frac{\chi(G)}{\log \chi(G)}\right)$?

Known for $G = K_{d+1}$.

Problem 2

Determine $\chi(L)$ for $G = K_5$.

(Known that a.a.s. $\chi(L) \in \{3, 4\}$.)

Conjecture

For any G, there is k_G with $\chi(L) = k_G$ a.a.s.

. . . and many more open questions!

Thank you!