Finite-size scaling for the random cluster model on random graphs

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Spin models on random graphs

Study statistical physics **spin models** on **random graphs**

Great source of **examples** and **counterexamples** in combinatorics

For probabilists, random graphs have non-trivial geometry but may be tractable due to connection with infinite trees (**Bethe lattice**)

A source of **hard computational problems** and gadgets in hardness reductions

Potts model

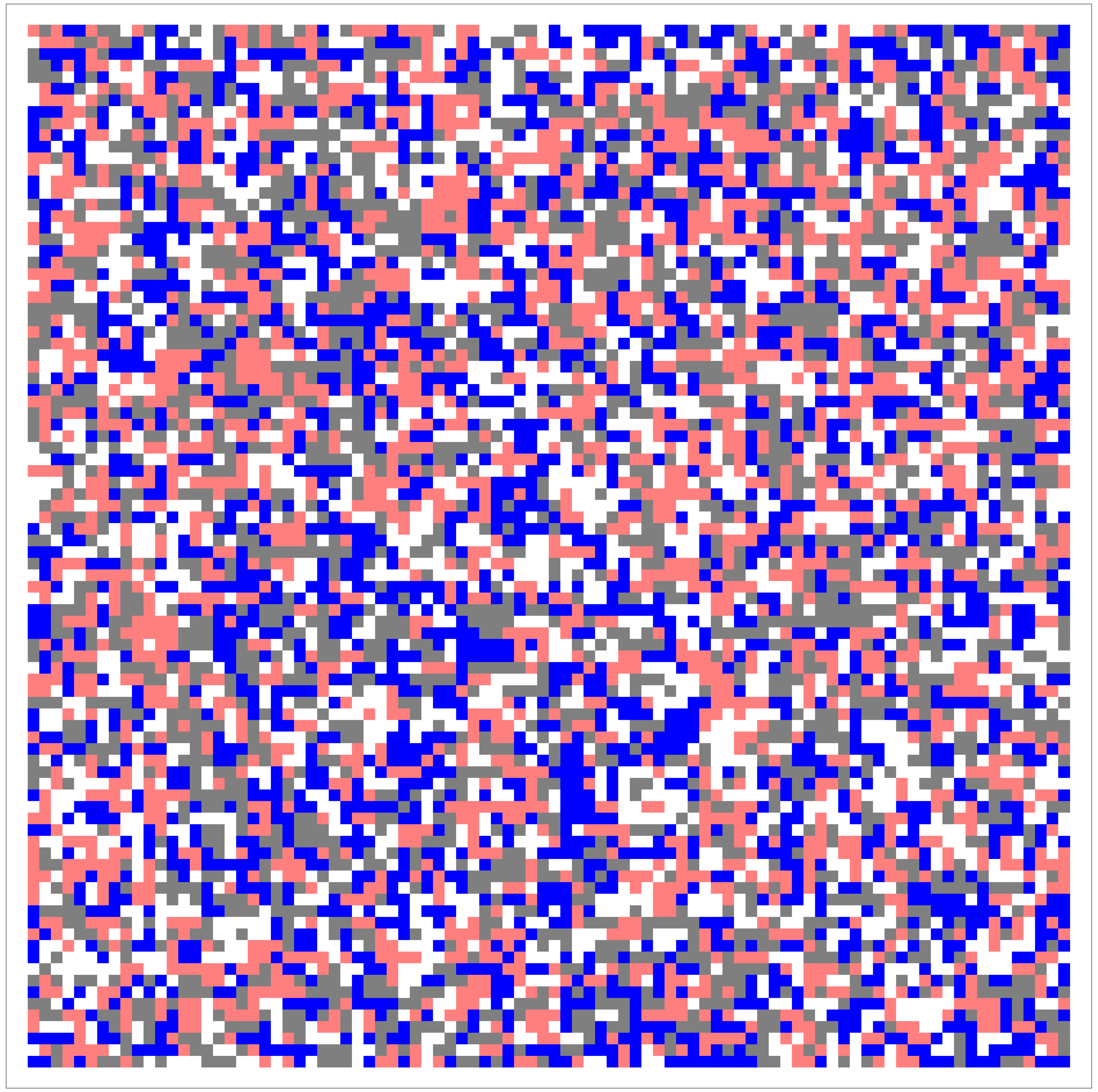
Probability distribution on assignments of q colors to vertices of G:

$$\mu_G^{\text{Potts}}(\sigma) = \frac{e^{\beta\, M(G,\sigma)}}{Z_G^{\text{Potts}}(q,\beta)} \qquad Z_G^{\text{Potts}}(q,\beta) = \sum_{\sigma} e^{\beta M(G,\sigma)}$$

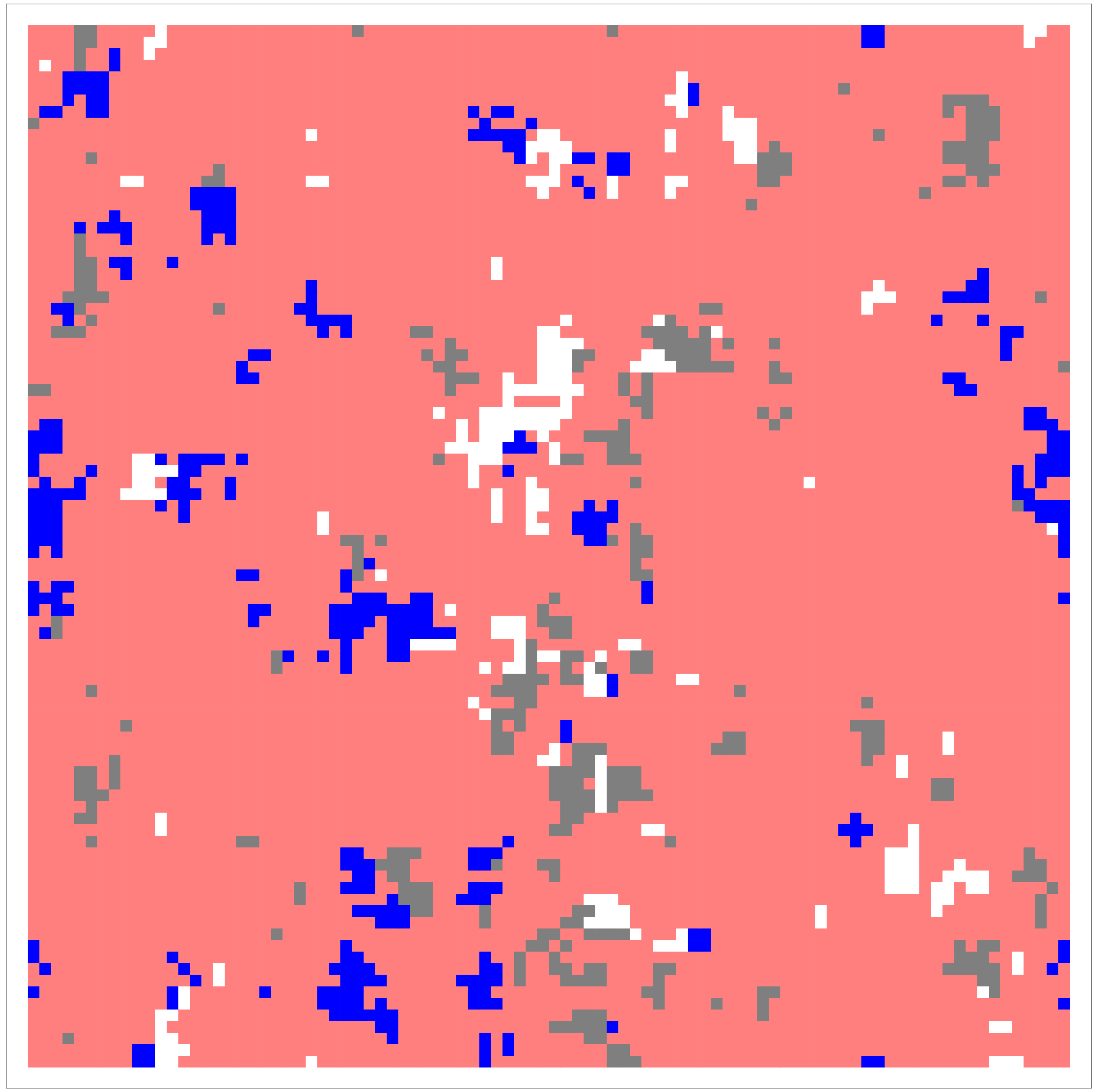
where $M(G,\sigma)$ is the number of **monochromatic edges**.

Inverse temperature $\beta > 0$ is the ferromagnetic case, $\beta < 0$ is anti-ferromagnetic.

Potts model



High temperature (β small)



Low temperature (β large)

Techniques to study spin models

In combinatorics: **first** and **second-moment methods**, **Azuma's inequality**, **Chernoff bounds**, **Friedgut**…

Give decent answers but fall short of pinning down thresholds (q-coloring threshold of G(n,p) still open!)

Non-rigorous cavity method from statistical physics predicts precise thresholds and structural changes (shattering, condensation,…).

Main objects: **Belief propagation**, **Bethe formula**, **replica symmetry breaking**

Many aspects of the cavity method have now been made rigorous: BP-inspired 2nd moment; interpolation method; **Coja-Oghlan**, **Ding-Sly-Sun**, **Mossel-Neeman-Sly**..

Our technique

Abstract polymer models and the **cluster expansion**

Classical technique from statistical physics: used e.g. by **Laanait-Messager-Miracle-Sole-Ruiz-Shlosman** to understand the large-q 1st order phase transition for Potts on $\mathbb{Z}^d$

Used recently in algorithms: joint w/ **Helmuth-P.-Regts**, **Jenssen-Keevash-P.**, **Borgs-Chayes-Helmuth-P.-Tetali**

Used recently in combinatorics: **Jenssen-P.**, **Balogh-Garcia-Li**, **Davies-Jenssen-P.**, **Jenssen-Keevash** (talks on the “Big Seminar” youtube channel)

Write a partition function in terms of defects from a ground state (**polymers**)

Express the log partition function as an infinite series (**sum over clusters**)

Potts model on random graphs

$\boxed{\text{Ferromagnetic Potts: always replica symmetric, q ground states}}$

Antiferromagnetic Potts: replica symmetry breaking, information-computation gap

Potts on Random Graphs

Ferromagnetic Potts model on random d-regular graphs (**Galanis-Stefankovic-Vigoda-Yang**; also **Dembo-Montanari-Sly-Sun**):

Let $\beta_c(q,d) = \log \dfrac{q-1}{(q-1)^{1-2/d}-1}$

For $\beta < \beta_c$, there is a **disordered phase** (marginals close to uniform whp)

For $\beta > \beta_c$, there are **q ordered phases** (marginals favor one dominant color)

For $\beta = \beta_c$, the q+1 phases each have at least $1/n^c$ probability (**weak phase coexistence**)

Good understanding of the **free energy**: $\lim_{n\to\infty} \dfrac{1}{n} \mathbb{E} \log Z_n$

Potts on Random Graphs

What precisely happens at $\beta = \beta_c$? (**phase coexistence**)

What is the distribution of $\log Z_n$? (**finite-size scaling**)

How do correlations behave? (**exponential decay of correlations**?)

What is the local spin distribution? (**local weak convergence**)

We can answer **all of these questions** in detail when $q = q(d)$ is large.
More generally, we do so for the **random cluster model**.

Random cluster model

Probability distribution on **subsets of edges** of G.

$$\mu_G(A) = \frac{q^{c(A)}(e^\beta - 1)^{|A|}}{Z_G(q,\beta)} \qquad Z_G(q,\beta) = \sum_{A \subseteq E} q^{c(A)}(e^\beta - 1)^{|A|}$$

where $c(A)$ is the number of **connected components** of (V, A).

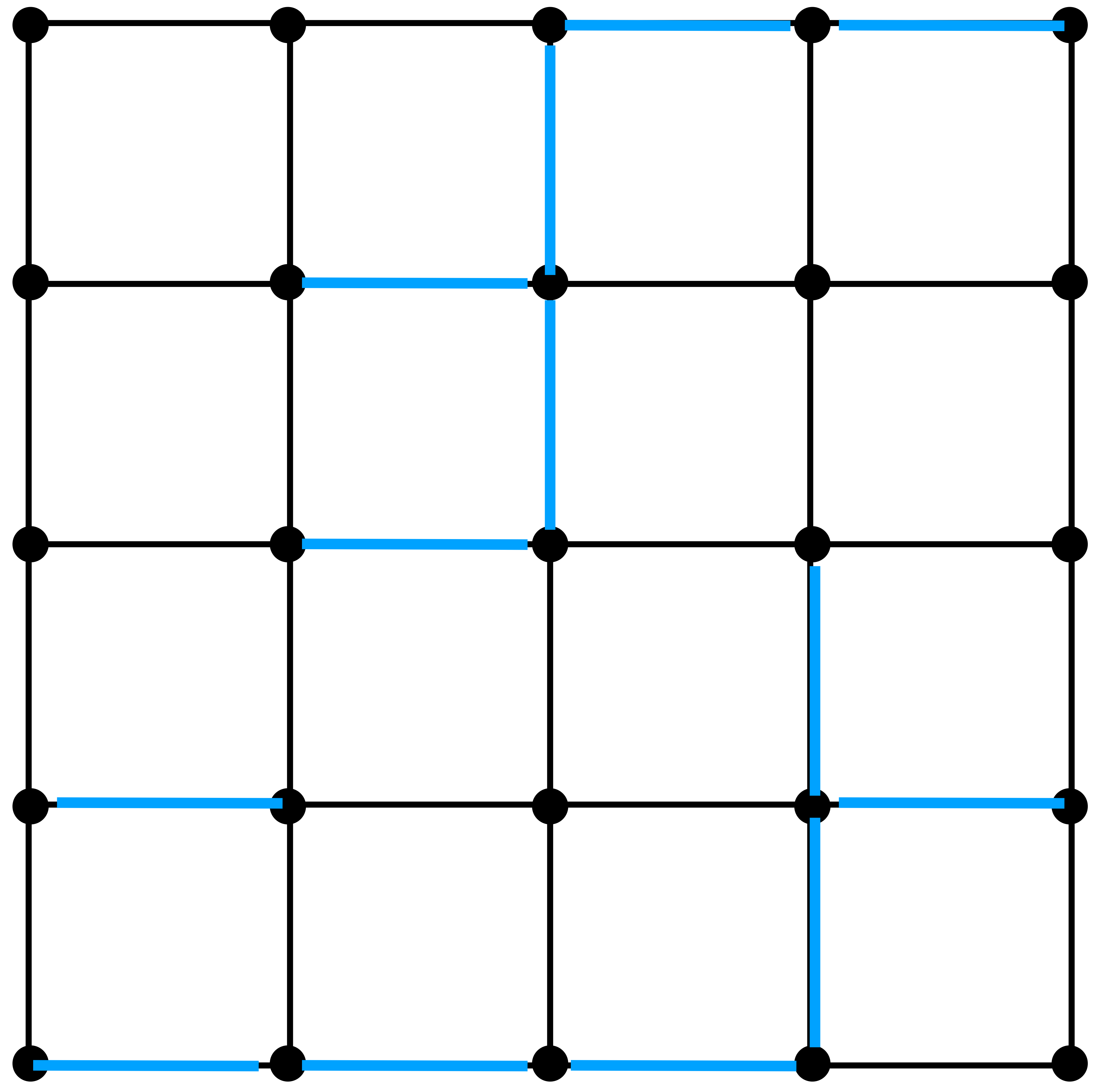
Two possible ground states: **disordered** $A = \emptyset$, **ordered** $A = E$.

$q > 0$ real

Edwards-Sokal coupling

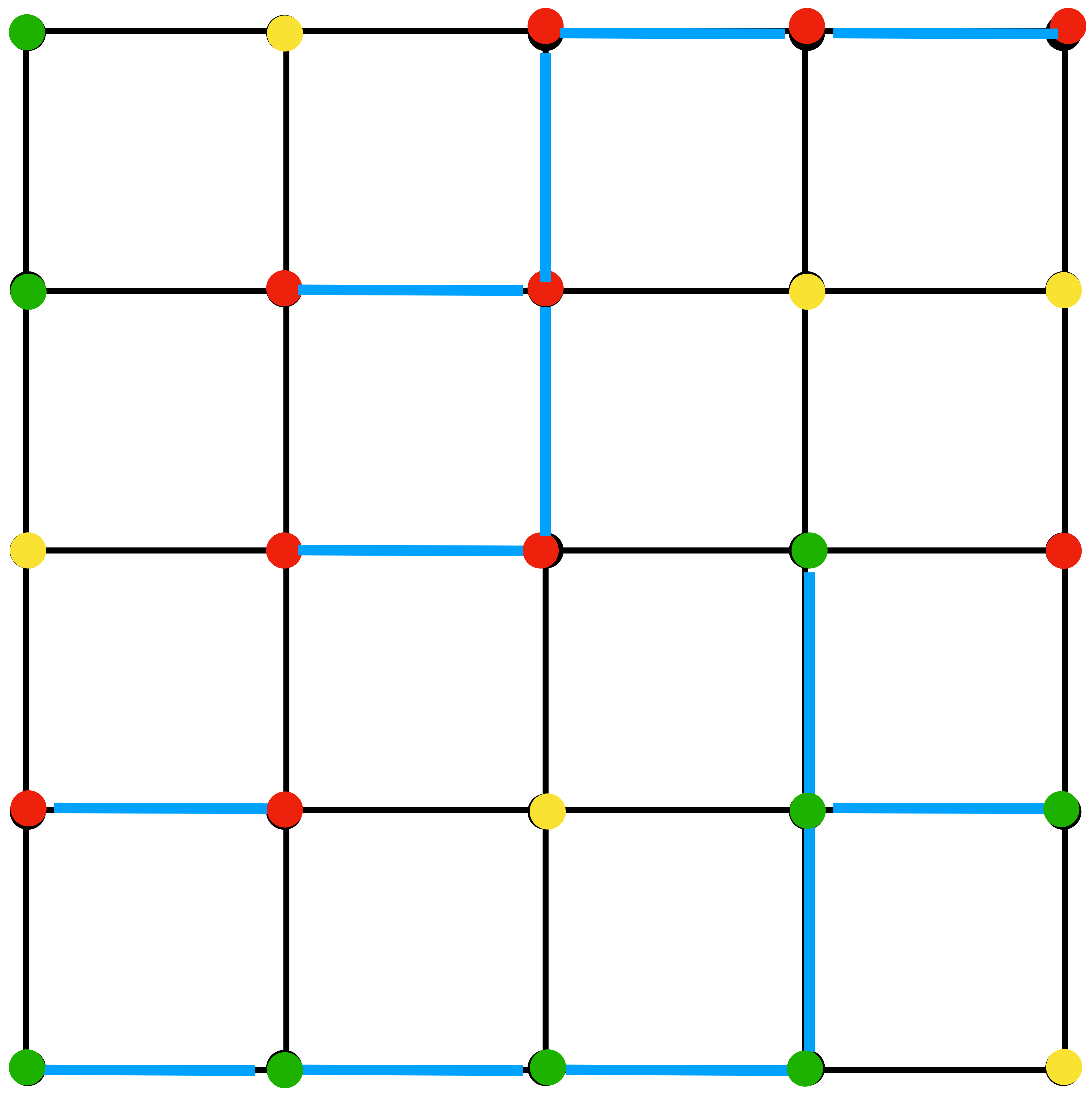
$$Z_G^{\mathrm{Potts}}(q,\beta) = Z_G(q,\beta)$$

1. Pick a set of edges according to the random cluster measure
2. Determine the connected components



Edwards-Sokal coupling

$$Z_G^{\text{Potts}}(q, \beta) = Z_G(q, \beta)$$



1. Pick a set of edges according to the random cluster measure
2. Determine the connected components
3. Assign one of the q colors uniformly and independently to each connected component

Main results

Thm. For $d \geq 5$ and $q = q(d)$ large enough, there exists $\beta_c(q, d)$ so that:

1. For $\beta \neq \beta_c$ the **free energy is analytic** and μ_n exhibits **exponential decay of correlations** whp.

2. μ_n **converges locally** to μ_{free} and μ_{wire} for $\beta < \beta_c$ and $\beta > \beta_c$ respectively

3. The **relative weights** of the ordered and disordered states at $\beta = \beta_c$ converge to given random variables (a function of small cycle counts). μ_n converges locally to a mixture of μ_{free} and μ_{wire}

Main results

Thm. For $d \geq 5$, $q = q(d)$ large enough, and **all** β there is an **FPTAS** and **efficient sampling algorithm** for the random cluster model on d-regular random graphs.

Algorithms work subject to **expansion conditions** that hold whp and can be checked efficiently.

Step 1: almost all or nothing

Use **expansion properties** to show that for q large, with probability $1 - \exp(-\Theta(n))$ a sample from the RC model consists of **at least .9** or **at most .1** fraction of edges. (Not hard)

Write $Z = Z_{dis} + Z_{ord} + Z_{err}$

Suffices to understand Z_{dis} and Z_{ord}, and μ_{dis} and μ_{ord}

Polymer models and cluster expansion

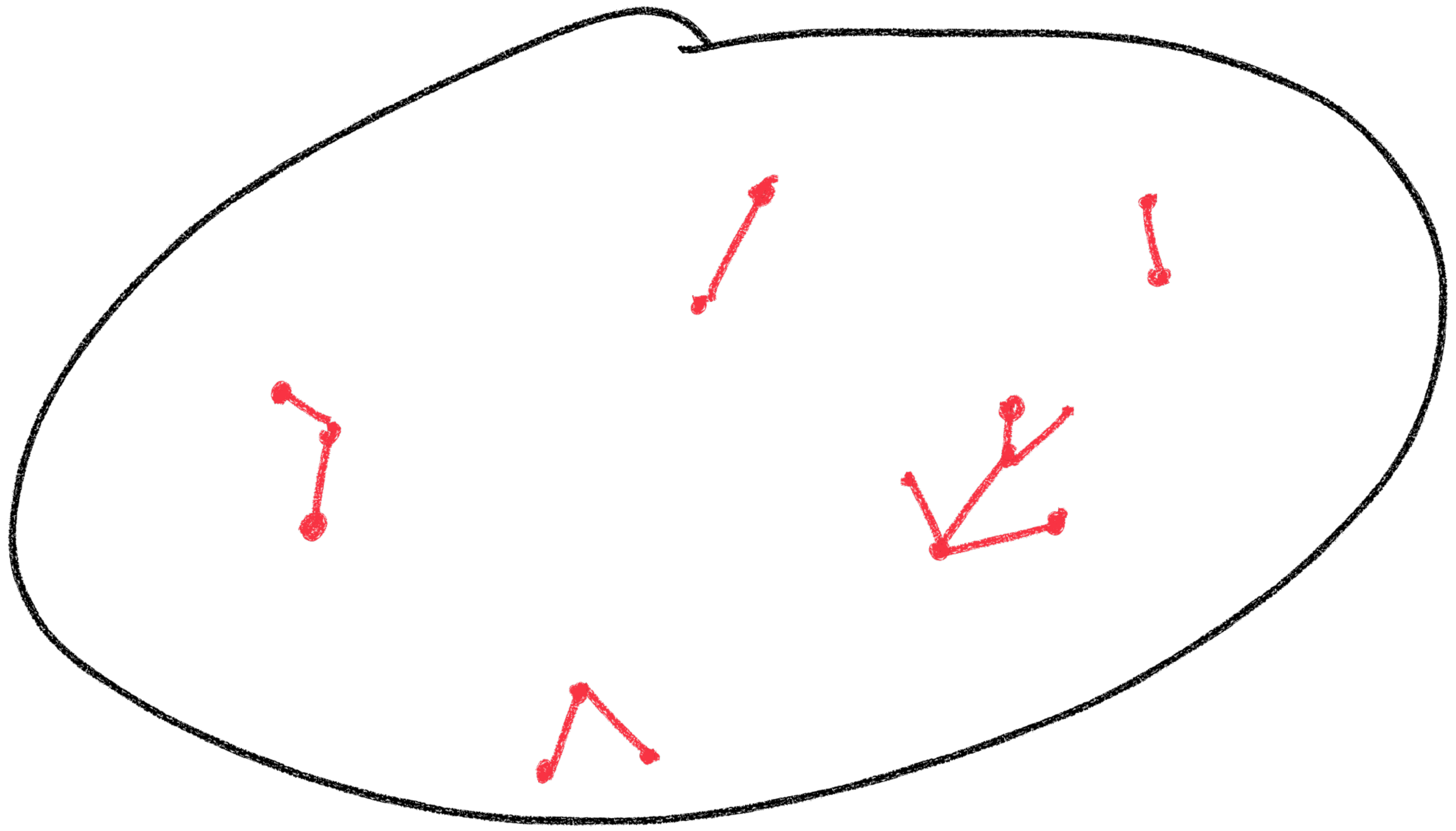
Rewrite partition function as a sum over collections of disjoint geometric objects (polymers) of product of polymer weights

If **weights decay fast enough** as a function of size, then the cluster expansion, a power series for log Z, converges

Weights must **factorize** and **decay**

Step 2: disordered

Express **disordered** configurations in terms of deviations from the **empty configuration**

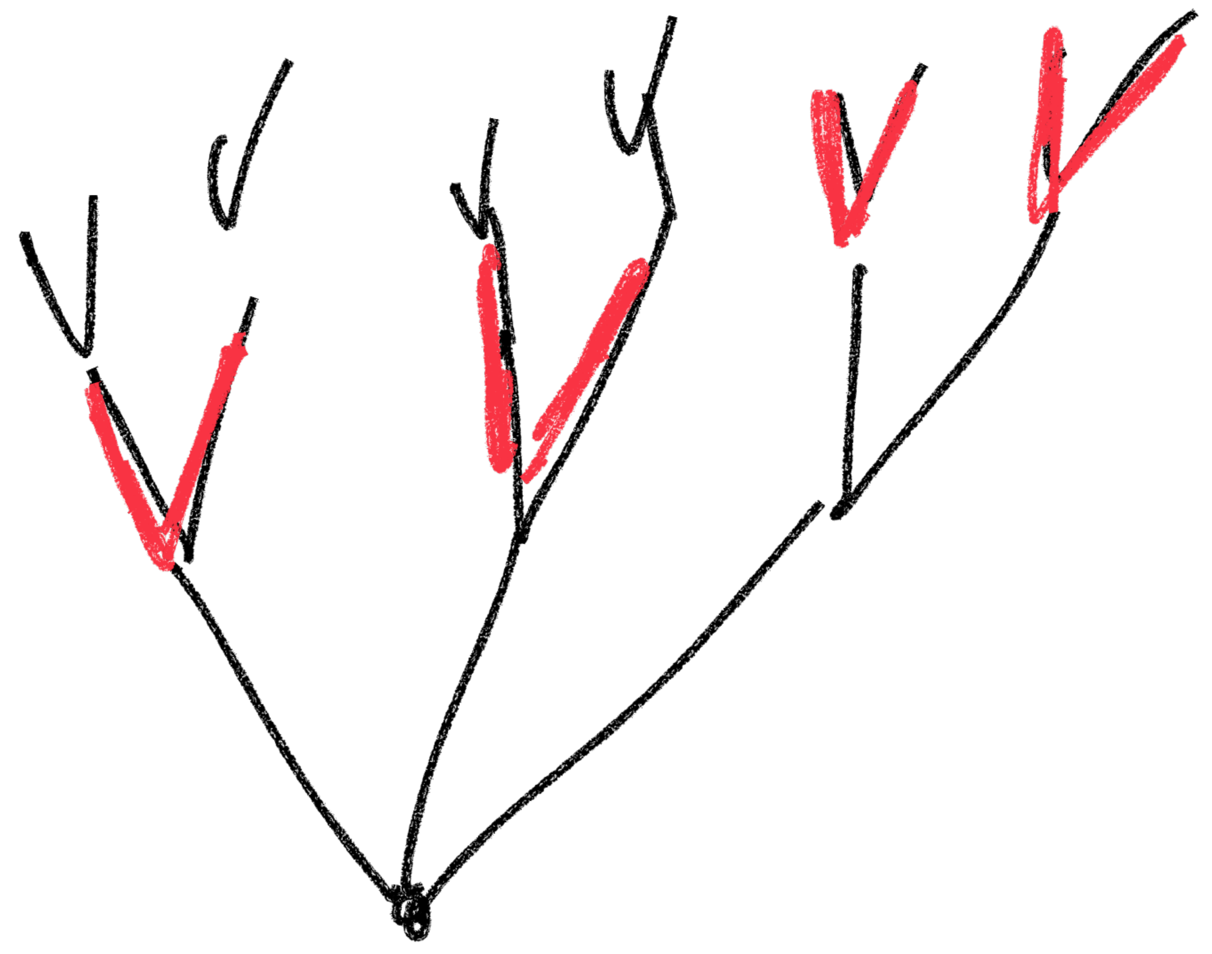


Polymers are **connected components of occupied edges**

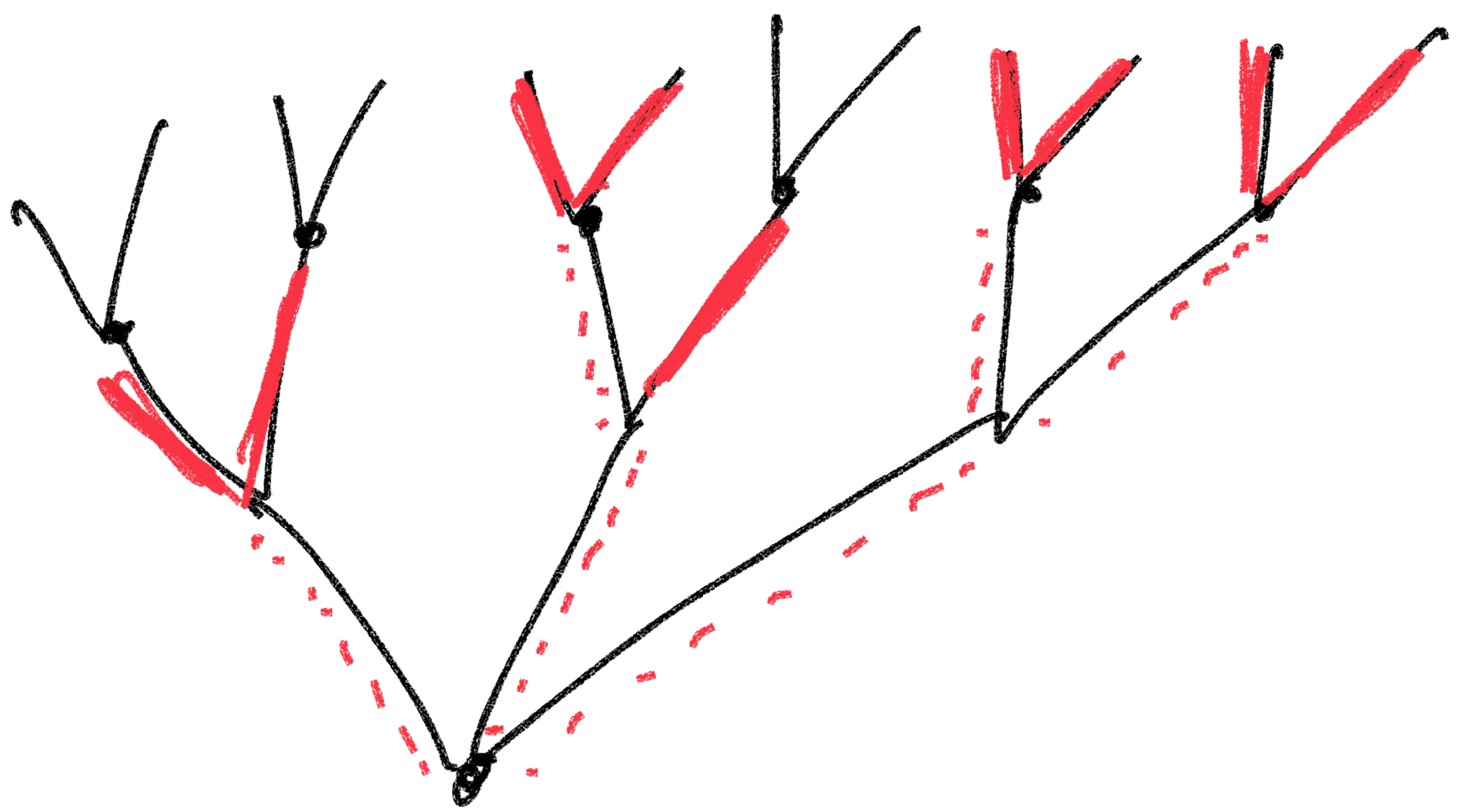
$$w(\gamma) = q^{1-|\gamma|}(e^{\beta} - 1)^{|E(\gamma)|}$$

Step 2: ordered

Express **ordered** configurations in terms of defects from the **all-occupied configuration**



Step 2: ordered



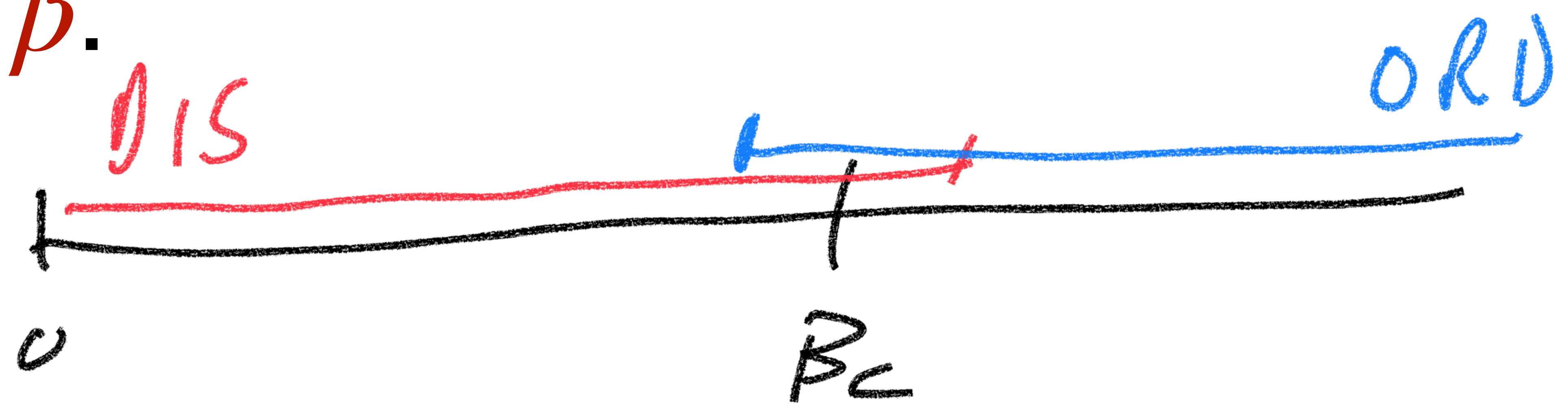
Define **boundary** by starting with unoccupied edges; inductively add all edges incident to any vertex with at least **5/9-fraction of its edges in boundary**.

Polymers are **connected components of the boundary**

$$w(\gamma) = q^{c'(\gamma)}(e^{\beta} - 1)^{-|E_u(\gamma)|}$$

Consequences

For q large, ordered and disordered cluster expansions converge in **overlapping range of β**.



Convergent cluster expansion gives properties like exponential decay of correlations, large deviation bounds, CLT’s…

This also gives algorithms at **all temperatures**.

Open questions

Prove that for the random cluster model on random d-regular graphs,

$$\beta_c = \log \frac{q-2}{(q-1)^{1-2/d}-1}$$

Extend the current results to all $d \geq 3$ (more refined def of ordered polymers)

Apply the second-moment method / cavity method to the **random cluster model**

Give sampling/counting algorithms for **hard-core on random bipartite graphs** for all λ

Other applications of **polymer models** and **cluster expansion** in probabilistic combinatorics

Thank you!