

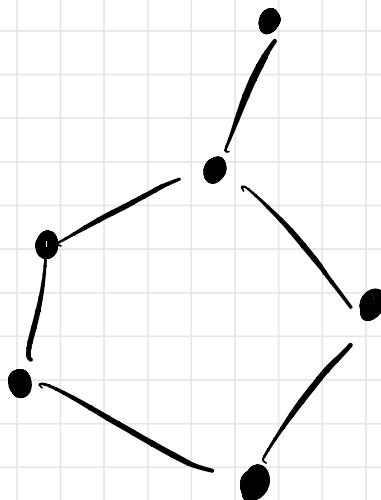
Localization Game for Random Graphs

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The game of Cops and Robbers

(Nowakowski, Winkler and Quilliot ; 1983 and 1978)



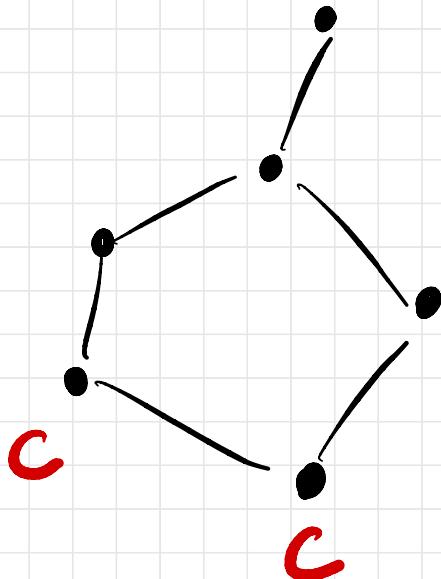
2 players :

- k cops
- 1 robber

The game of Cops and Robbers

(Nowakowski, Winkler and Quilliot ; 1983 and 1978)

$$k=2$$



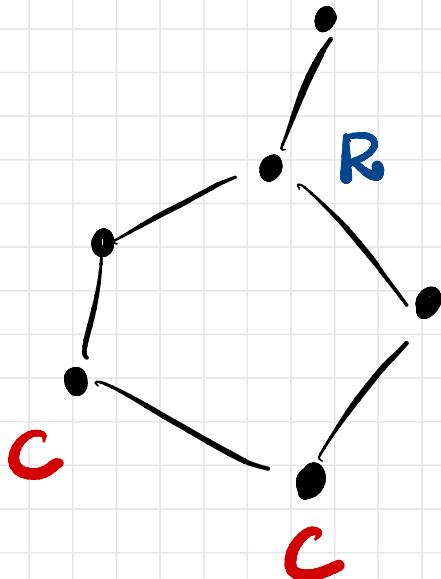
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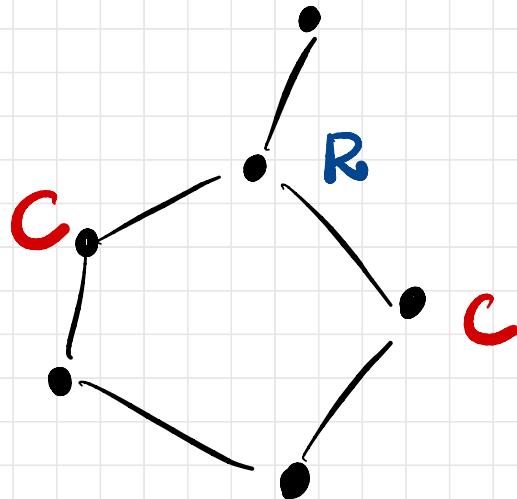
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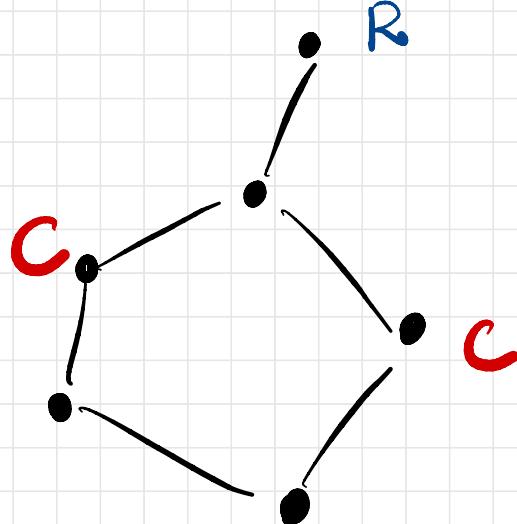
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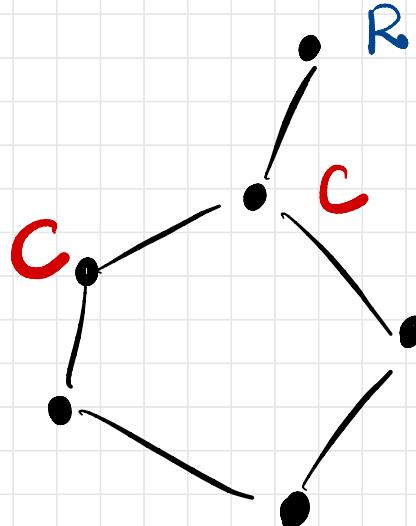
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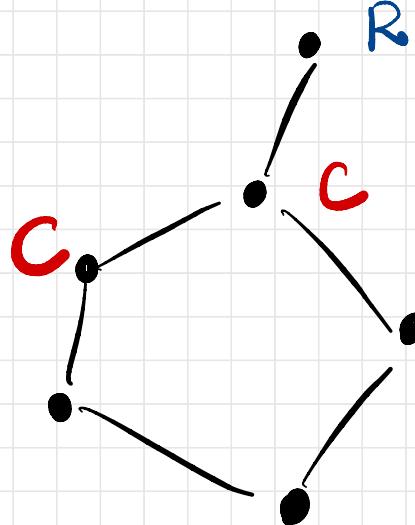
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The game of Cops and Robbers

(Nowakowski, Winkler and Quilliot ; 1983 and 1978)

$$k=2$$



2 players :

- k cops
- 1 robber

$$c(G) = \min \# \text{ of cops to win the game}$$

Meyniel's Conjecture

(communicated by Frankl, 1987)

$c(n) = O(\sqrt{n})$, where $c(n)$ is the maximum $c(G)$ over all connected graphs on n vertices.

- best possible
- wide open

Frankl, 1987 : $O\left(\frac{n \ln \ln n}{\ln n}\right)$

Chiniforooshan, 2008 : $O\left(\frac{n}{\ln n}\right)$

Lu, Peng, 2012

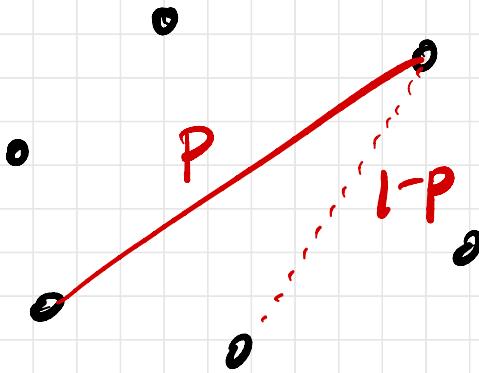
Scott, Sudakov, 2011

Friese, Krivelevich, Loh, 2012

$$\begin{aligned} & \left. \begin{aligned} & \leq n^{2 - (1+o(1)) \sqrt{\log_2 n}} \\ & = n^{1 - o(1)} \end{aligned} \right\} \end{aligned}$$

Binomial Random Graphs

$G(n, p)$



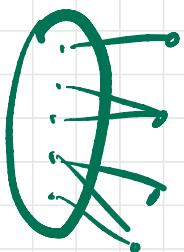
Event holds **asymptotically almost surely (a.a.s)**
if it holds with probability $\rightarrow 1$ as $n \rightarrow \infty$.

Thm : Bonato , Prabat , Wang ; 2009

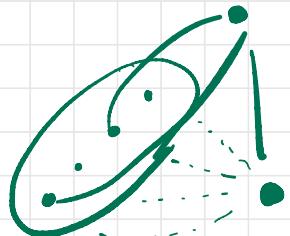
$$\frac{2\sqrt{n} \log n}{n} \leq p \leq 1 - \varepsilon, \text{ a.a.s.}$$

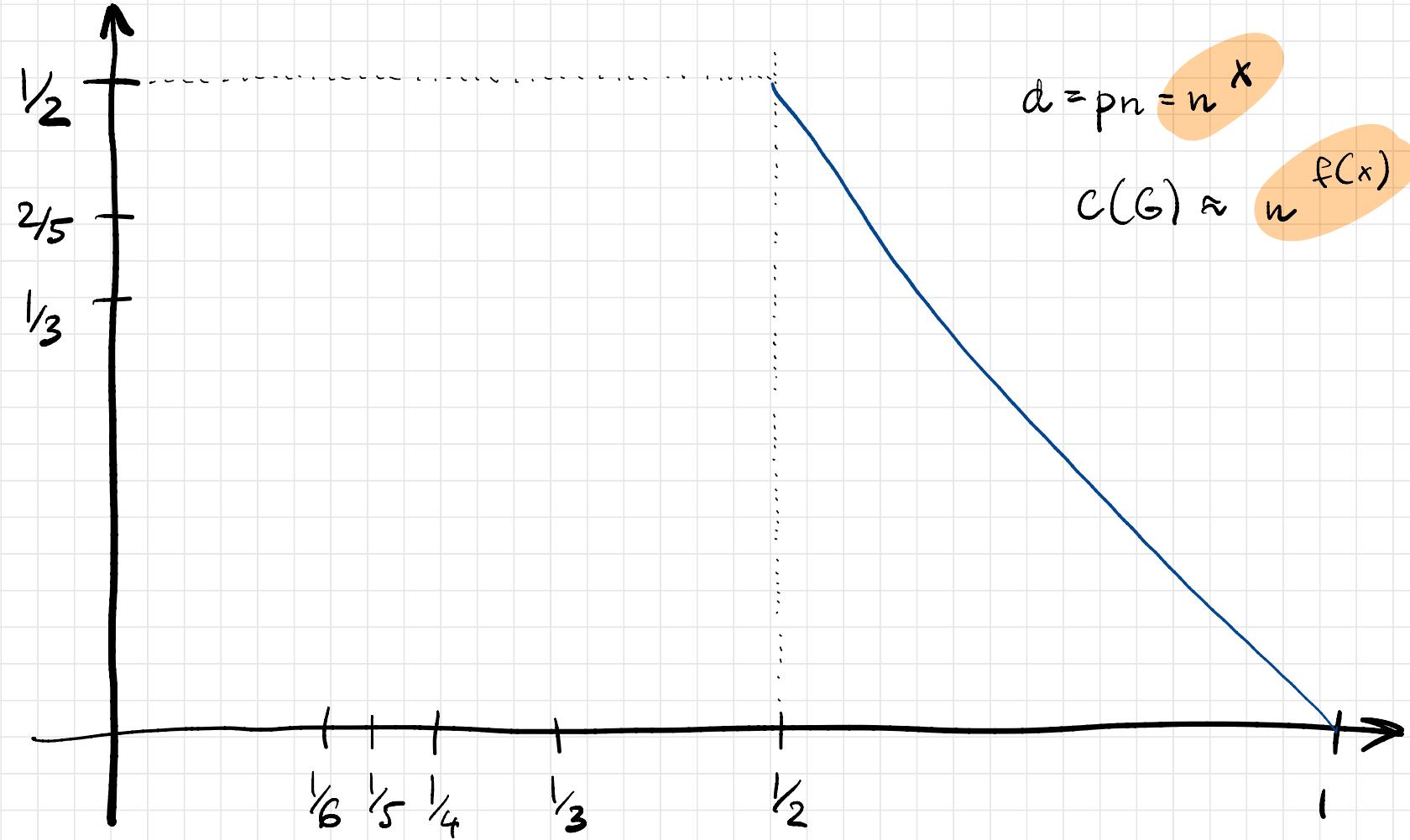
$$\log_{\frac{1}{1-p}}(n) - \log_{\frac{1}{1-p}}\left(\frac{\log_{\frac{1}{1-p}}(n) \cdot \log(n)}{p}\right) \leq c(G(n, p)) \leq \log_{\frac{1}{1-p}}(n) + \log_{\frac{1}{1-p}}(\omega)$$

Upper bound : (trivial) $c(G) \leq \delta(G)$



Lower bound : $(1, k)$ -e.c.

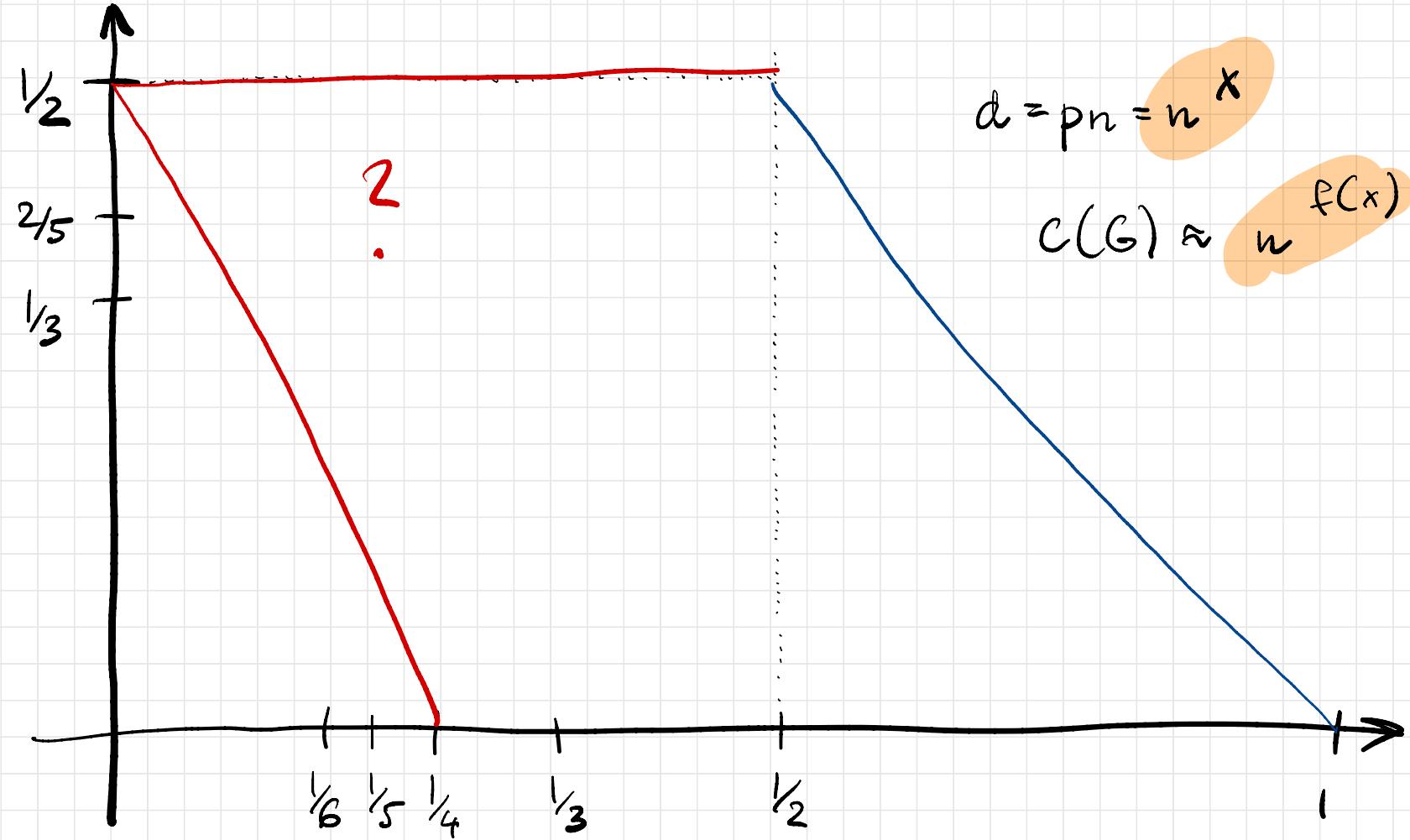




Thm : Bollobás, Kun, Leader , 2013

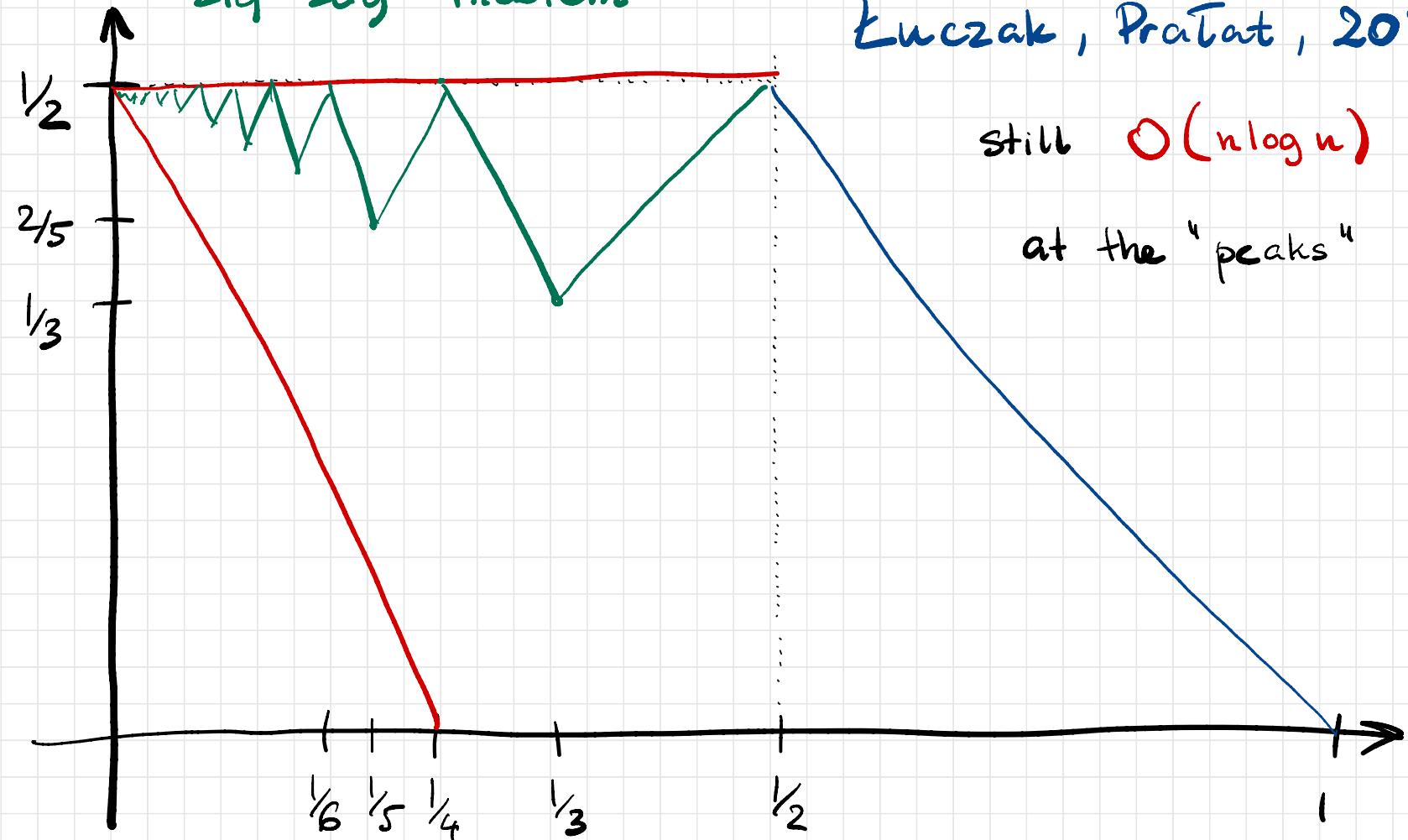
$$p \geq \frac{2.1 \log n}{n}, \text{ a.a.s}$$

$$\frac{n^{1/2 + o(1)}}{(np)^2} \leq c(G(n,p)) \leq O(\sqrt{n} \log n)$$

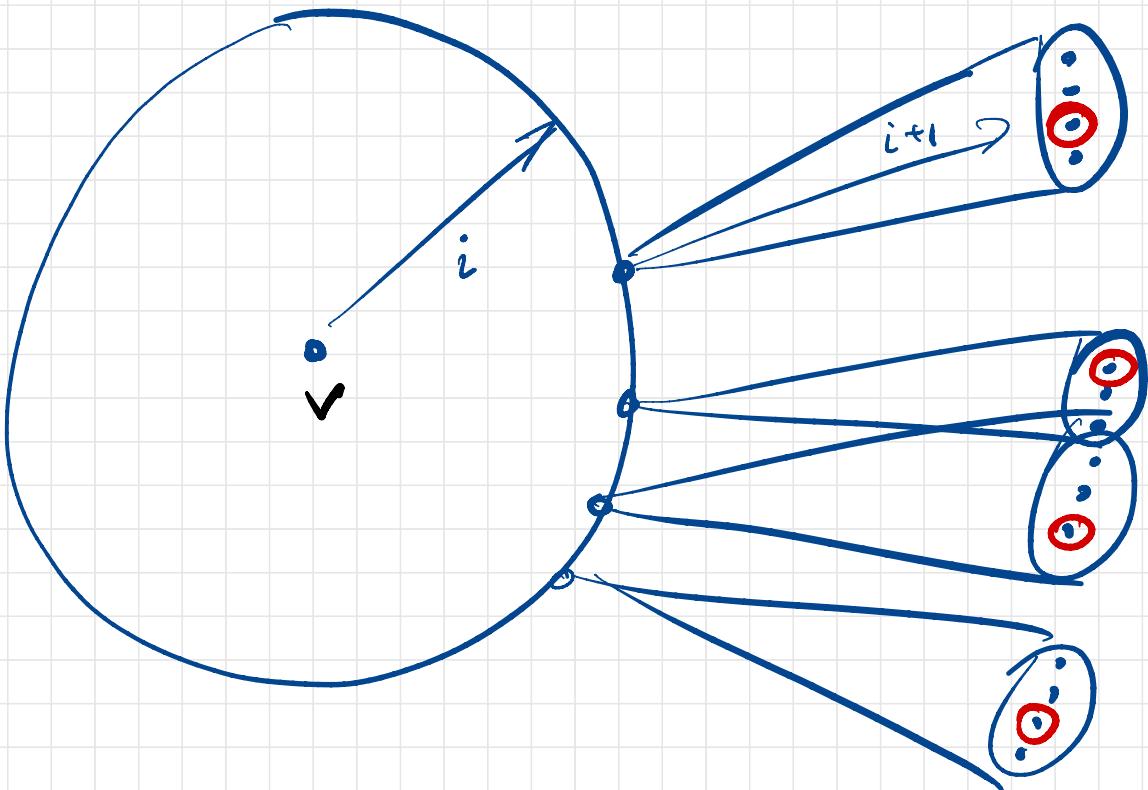


"Zig-zag" theorem

Kuczak, Pratap, 2010



Find a perfect
matching
between
 $s(v, i)$
and
(a subset of)
the cops



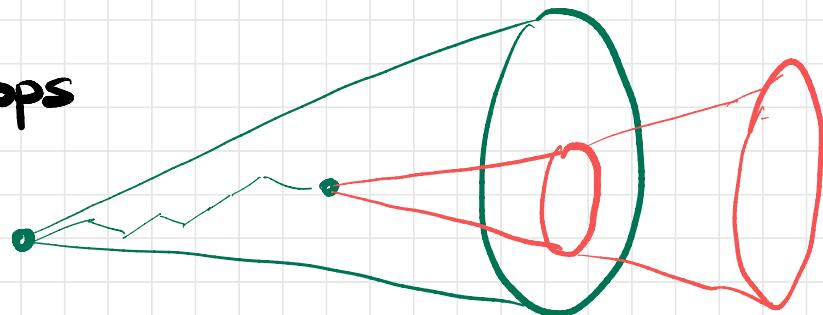
Thm : Pratet , Wormald , 2016

$$pn \geq (\gamma_2 + \varepsilon) \log n, \text{ a.a.s. } c(G(n,p)) = O(\sqrt{n})$$

Thm : Pratet , Wormald , 2019

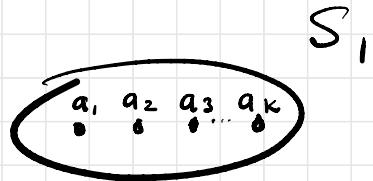
$$d = d(n) \geq 3, \text{ a.a.s. } c(\text{rand. } d\text{-reg}) = O(\sqrt{n})$$

$\log \log n$ ind. teams of cops
 $\log \log n$ rounds



Localization Game with k sensors

- G is known to both players
- The cops choose $S_1 \subseteq V(G)$ of size k
- Invisible robber chooses $v \in V(G)$



and announces $(\text{dist}(v, a_1), \dots, \text{dist}(v, a_k))$

S_1 -signature \rightarrow

$v \cdot (5, 3, 2, \dots)$

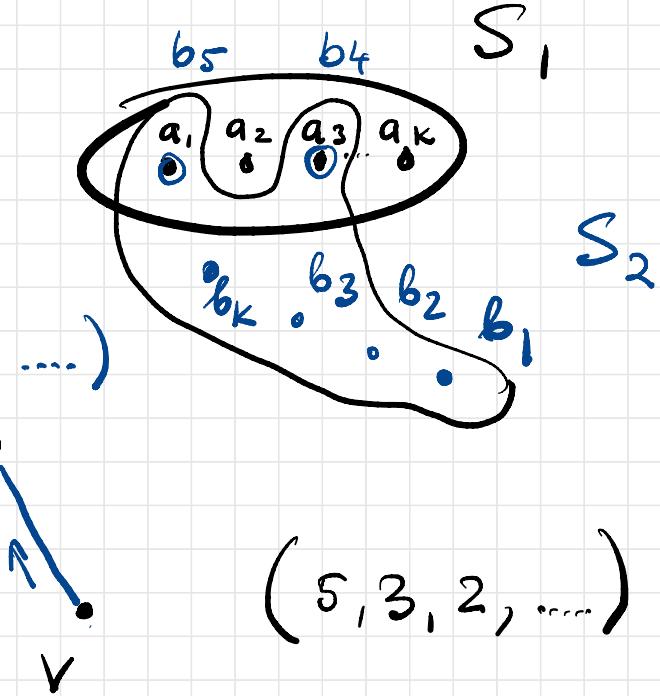
The game ends if this is sufficient to locate the
(cops win) \rightarrow robber.

Otherwise, the game continues ...

... if this additional info is enough to locate the robber, the cops win.

$(2, 3, 1, \dots)$

N



Being invisible is tricky.

Consider two robbers

Alice , Bob , playing
the same game.

Cops

Alice / Bob

s_1

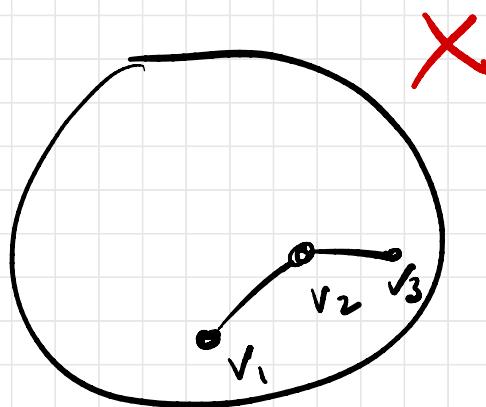
(5, 3, 8, ...)

s_2

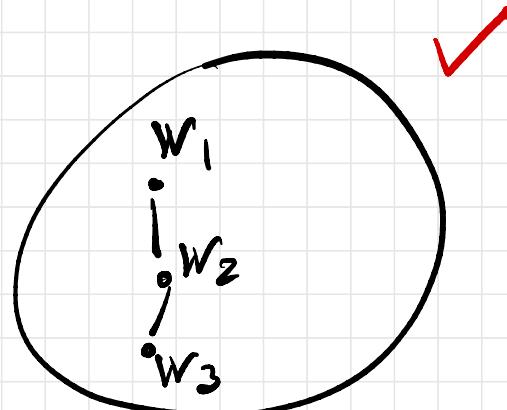
(1, 3, 7, ...)

s_3

(4, 4, 4, ...)

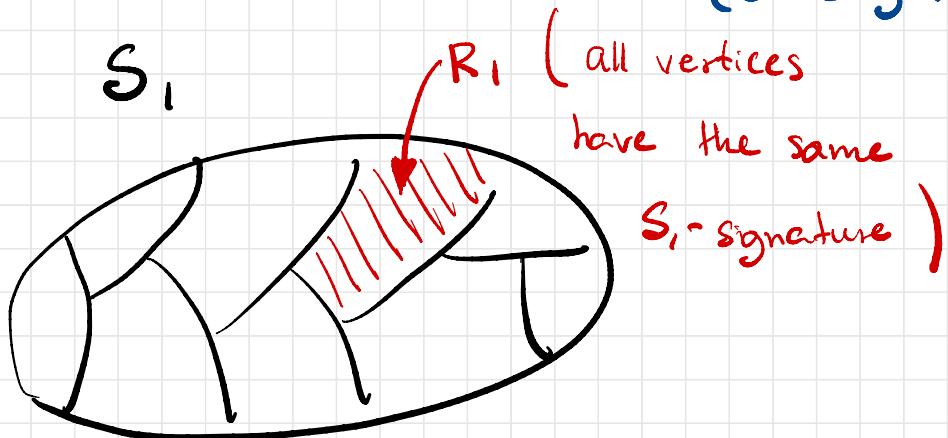


Alice

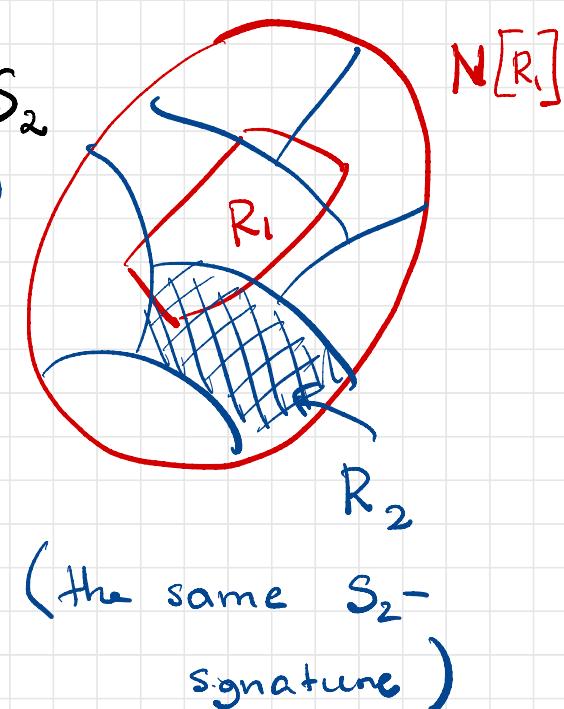


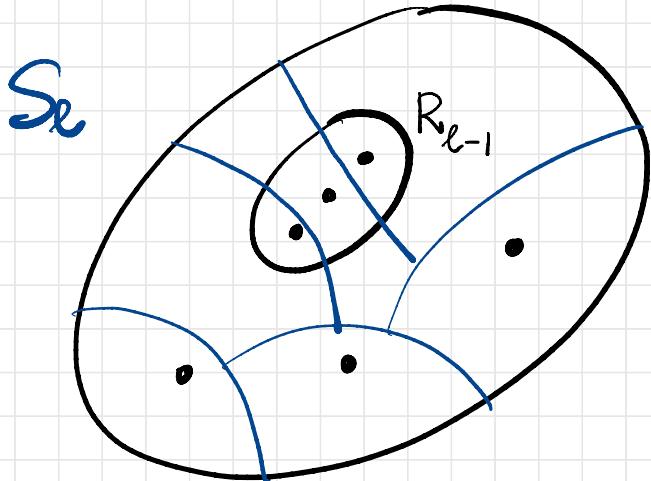
Bob

- Consider worst case scenario (for cops)
- play against many players at the same time
- cops provide their strategy up front
- the robber can "teleport"
- use the "cops' point of view"



(comb. game)





Cops win if they have a strategy to reach such state.

Otherwise, the robber wins.

$\ell(G) = \min$ # of sensors (k) so that the cops can eventually locate the robber.

($k=1$) Seager, 2012

Is there a trivial upper bound s.t. $c(G) \leq \beta(G)$?

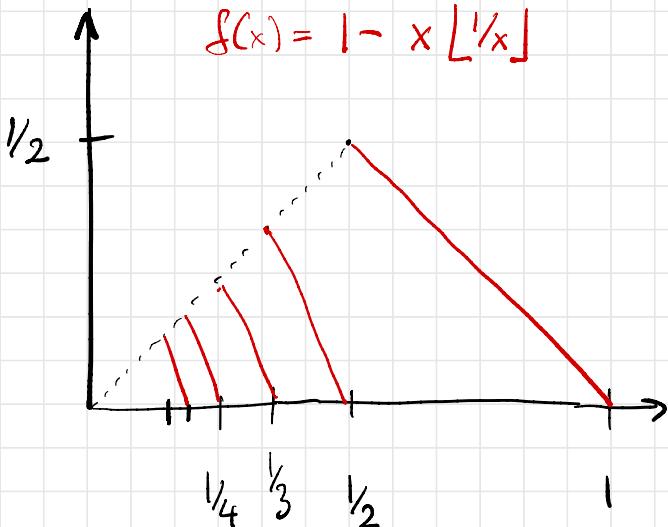
Yes! $\ell(G) \leq \beta(G)$ ← metric dimension

(the cops win in one round)

$\beta(G) = \min k$ s.t. there exists a set S of size k
s.t. all vertices have unique S -signature.

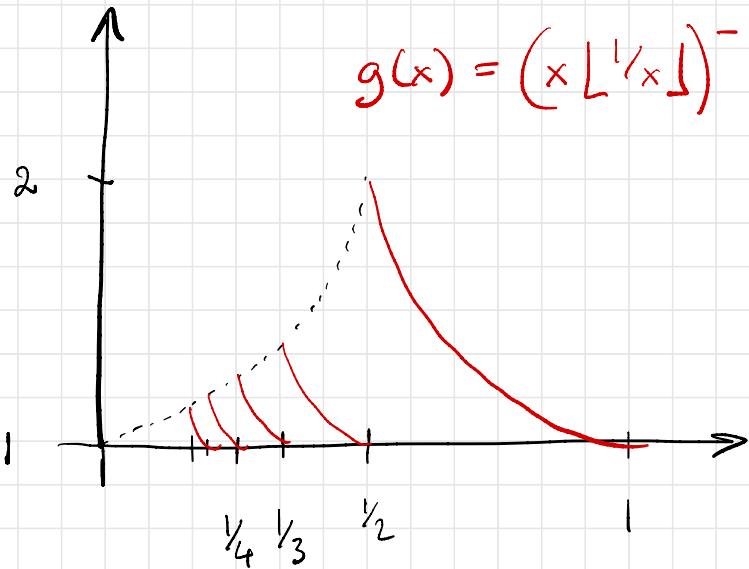
Slater, 1975

Harary, Melter, 1976



$$d = pn = n^{\alpha}$$

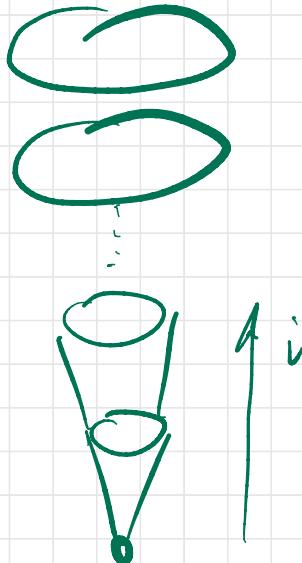
$$\beta \approx n^{f(x)}$$



the ratio
between
upper / lower
bounds

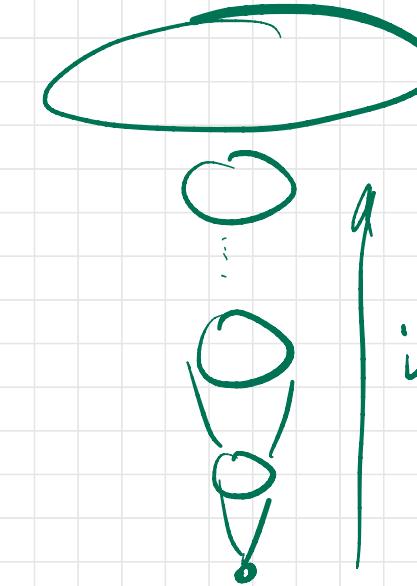
Bollobás, Mitsche, Prałat, 2013

$$\beta \approx \frac{n \log n}{d^i}$$



distinguishes

$\sim n^2/4$
pairs



$$d^i = o(n)$$

$\sim d^i n$ pairs

Localization number for $G(n, p)$ - diameter 2

Dudek, Frieze, Pegden, 2019

$$f(x) = \begin{cases} x, & x \in (2/3, 1) \\ 1-x/2, & \text{otherwise} \end{cases}$$

$$d = pn = n^{x+o(1)}, \quad x \in (1/2, 1)$$

$$(1+o(1)) (2x-1) \frac{n \log n}{d} \leq \ell(G) \leq (1+o(1)) f(x) \frac{n \log n}{d}$$

$$d = pn = n^{1+o(1)}$$

$$\ell(G) = (1+o(1)) \frac{2 \log n}{\log(1/\varsigma)}, \quad \varsigma = p^2 + (1-p)^2$$

More work left :

- determine the constant
- investigate sparse graphs.
- is $\ell(G)$ bounded away from $\beta(G)^2$?

Localization number for $G(n, p)$ - diameter $i+1$

Dudek, English, Frieze, MacRury, Pralet, 2020+

diameter $i+1$: $d^{i+1}/n - 2 \log n \rightarrow \infty$

"not too close" to i : $d^i = o(n)$

$$l(G) \sim \frac{n \log d}{d^i} \quad (d \geq (\log n)^\omega)$$

- $d^i = c \in \mathbb{R}_+$

$$l(G) = \Theta\left(\frac{n \log d}{d^i}\right) \quad (d > \log^3 n)$$

- $d^i = c(n) \rightarrow \infty$

$$\frac{\text{Upper}}{\text{Lower}} \rightarrow \infty$$

In particular, $d = n^{x+o(1)}$, $x \in \left(\frac{1}{i+1}, \frac{1}{i}\right)$, $i \in \mathbb{N}$

$$(1+o(1)) \frac{n \log(d^i)}{d^i} \leq \beta(G) \leq (1+o(1)) \frac{n \log n}{d^i}$$

$$l(G) \sim \frac{x n \log n}{d^i}$$

Hence,

$$i + o(1) \leq \frac{\beta(G)}{l(G)} \leq \frac{1}{x} + o(1) < i + 1$$

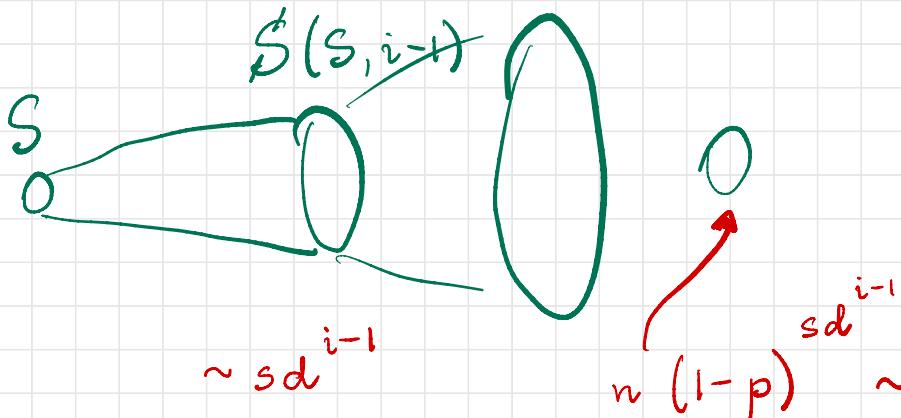
We get separation if $i \geq 2$ ($\text{diam} \geq 3$)

Lower bound : $s = (\log d - 3 \log \log n) \frac{n}{d^i}$

Strategy for the robber : stay with signature $(i+1, i+1, \dots, i+1)$

Observation 1 : $\forall S \subseteq V, |S|=s, |\{v \in N(S, i)\}| \geq r/2,$

where $r = \frac{n \log^3 n}{d}$



Union bound :

$$\binom{n}{s} \exp(-\Theta(r)) \rightarrow 0$$

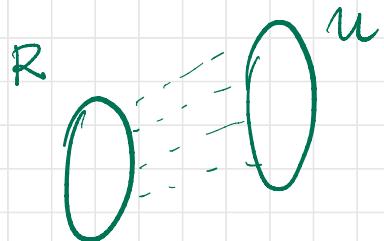
$$n (1-p)^{sd^{i-1}} \sim n \exp\left(\frac{-sd^i}{n}\right) \sim \frac{n \log^3 n}{d} = r$$

The robber can follow her strategy in the first round.

Can she continue ?

Observation 2 :

$$\forall R \subseteq V \quad |R| = \lceil r/4 \rceil \quad |\text{VAN}(R, i)| \leq \lceil r/4 \rceil$$



$$\begin{aligned} P(|U| \geq \lceil r/4 \rceil) &\leq \binom{n}{\lceil r/4 \rceil} (1-p)^{|R| \cdot \lceil r/4 \rceil} \\ &\leq \left(\frac{4ne}{r}\right)^{\lceil r/4 \rceil} \exp\left(-\frac{d}{n} \cdot \frac{r}{4} \cdot \frac{r}{4}\right) \\ &\leq \exp\left(\frac{r}{4} \cdot \log n - \frac{r}{16} \log^3 n\right) \\ &= \exp(-\Theta(r \log^3 n)) \end{aligned}$$

$$\text{Union bound : } \binom{n}{\lceil r/4 \rceil} \exp(-\Theta(r \log^3 n)) \rightarrow 0$$

Upper bound : $k \sim (\log d + 2 \log \log n) \frac{d}{d}$

Prove deterministic bound for a large enough graph that

- $|V| = n$ satisfies
- the diameter is $i+1$
- $x \neq y, 1 \leq j \leq i \quad |S(x, j) \setminus S(y, j)| \sim d^j$
- $\Delta \leq (1 + o(1))d$

$G(n, p)$ satisfies these properties.

Proof : non-constructive (the cops play randomly)

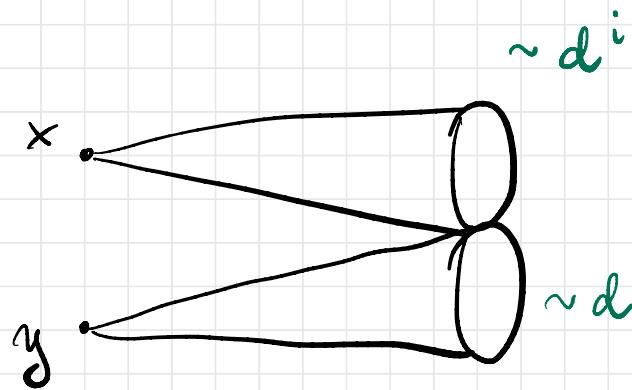
$$|N(R_1), 1| \leq n, \quad |N(R_{t+1}), 1| \leq |N(R_t), 1| \quad \frac{1}{\log n} \leq t \leq \log n / \log \log n.$$

There are

$$\binom{|N(R_{t+1})|}{2}$$

pairs of vertices.

Consider one of them



Prob no sensor is placed in
[$S(x,i) \setminus S(y,i)$] \cup [$S(y,i) \setminus S(x,i)$]

$$= \left(1 - \frac{2d^i}{n}\right)^k \sim \exp\left(-\frac{2d^i}{n} k\right)$$

$$= \frac{1}{d^2 \log^4 n}$$

$$\mathbb{E}(\# \text{ of pairs in the same equivalence class}) \leq \frac{|N(R_{t+1})|^2}{d^2 \log^4 n}$$

$$\text{Markov's : } \# \leq \mathbb{E}[\#] \cdot \log n \leq \frac{|N(R_{t+1})|^2}{d^2 \log^3 n}$$

$$\text{No class} \geq 2\sqrt{\#}$$

$$|R_{t+1}| \leq \frac{|N(R_{t+1})|}{d \log n}$$

Future directions :

- random d -reg graphs ?
- random geometric graphs ?
($\beta(G)$ seems to be challenging
 $\ell(G)$ might be easier)
- Meyniel's conjecture for $c(G)$